SOLVED PAPER

M.Sc. MATHEMATICS (Part II)

MMP

SARFRAZ PHOTOCOPY

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INITIAL, BOUNDARY AND EIGEN VALUE Poblems

D-E and their Classifcation

Definition:- An equation involving one dependent variable and its derivatives with respect to one or more independent variables, is called "Differential Equation."

For Example:

(i) $\frac{dy}{dx} + y\sin x = \cos x$

(ii) $\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^3 = 0$

(iii) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = \frac{d^2y}{dx^2}$

(iv) $x\frac{\partial z}{\partial x} + y\,\partial z/\partial y = nz$

(v) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

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Ordinary Differential Equation

Definition:- A differential equation, in which ordinary derivatives of the dependent

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variable with respect to more than one independent variable occur, is called an "Ordinary Differential Equation".

Examples :- (i) $\frac{dy}{dx} + y^3 \cos x = \sin x$

(ii) $\frac{d^2y}{dx^2} + x^2 y^3 \left(\frac{dy}{dx}\right)^5 = 0$

(iii) $\left[2 + \left(dy/dx\right)^2\right]^{5/2} = \frac{d^2y}{dx^2}$

P-D-E (Partial Differential Equation)

Definition :-

A differential equation involving partial derivatives of the dependent variables with respect to more than one independent variable is called "Partial Differential Equation".

Examples :-

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Order and Degree of D-E

Order : The order of a differential equation is the order of the highest derivative that occures in equation

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Degree :- The degree of a differential equation is the greatest exponent of the highest order derivative that appears in the equation. (The dependent variable and its derivatives should be expressed in a form free from radicals and fractions).

Examples :-

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(1) D-E $y \cdot \cos x + \frac{dy}{dx} = \sin x$

has order 1 and degree 1

(2) $\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0$

order = 2 and degree = 1

(3) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = \frac{d^2y}{dx^2}$

order = 2 & degree = 2

Exponent of $\frac{d^2y}{dx^2}$ is 2 after removing the radical by squaring both side of the equation.

(4) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx$

order = 1 and degree = 1

(5) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

order = 2 and degree = 1

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Homogenous D-E

A differential equation for which $u=0$ is a solution is called a homogenous D-E where u is the unknown function.

In other words, a D.E which always possesses the trivial solution $u=0$ is called homogenous Differential Equation.

Example:- $\frac{d^2u}{dx^2} + x\frac{du}{dx} + 8u = 0$ is a homogenous D-E

and as example of non-homogenous D-E we have

$$a\frac{d^2u}{dx^2} + b\frac{du}{dx} + cu = \phi(x)$$

Similarly an Example of a homogenous PDE is provided by $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Linear D-E

Ä¡, ODE of order "n" $F(x, y, y', y'', y''', \text{----} y^{(n)}) = 0$ is said to be linear if F is a linear function of the variables $y, y', y'', y''' \text{---} y^{(n)}$.

In other words if the dependent variables and its derivatives occur only linearly in the equation, then it is linear. otherwise it will be nonlinear.

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Examples:- $\frac{d^2y}{dx^2} + y = yx^{12}$ (Linear D-E)

and the equation $\frac{d^3y}{dx^3} + 3e^x\frac{d^2y}{dx^2} + y\cdot\frac{dy}{dx} = x^3$

is not linear because the term $y\cdot\frac{dy}{dx}$

Note:- It should be carefully noted that in a linear ordinary D-E.

(i) The dependent variable y and its derivative are all degree one.

(ii) No products of y or any of its derivative appear

(iii) No transcendental function of y and or its derivatives appear.

Initial and Boundary Conditions

Initial Condition:-

It is often required to find the solution of a differential equation subject to certain conditions. If the conditions relate to one value of the independent variable such as $y(x_0) = y_0$ and $\frac{dy}{dx} = y'(x_0)$ at $x = x_0$ where x_0 belongs to some interval $]\alpha, \beta[$ then they are called initial conditions (or one-point boundary conditions) and x_0 is called initial point.

Initial value Problem

An initial value problem consists of D-E

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(of any order) together with a collection of initial conditions that must be satisfied by the solution of the differential equation and derivatives at the initial point.

Boundary Condition:-

The problem of finding the solution of a D-E such that all the associated constrains relate to two different values of the independent variable is called a two-point boundary value problem or simply a boundary value problem. The associated supplementary boundary conditions are called two point boundary conditions.

The boundary conditions are usually of the form

$$y(a) = y_1 \quad , y(b) = y_2$$

or

$$y'(a) = y_1 \quad , y'(b) = y_2$$

Examples 1:- Solve initial value problem

$$y' = 2x \quad , y(1) = 4$$

Solution:- $y = x^2 + C$, C being arbitrary constant

using the initial condition

$$y(1) = 1^2 + C$$

But $y(1) = 4$

$$4 = 1 + C$$

$$\boxed{C = 3}$$

Thus $y = x^2 + 3$ is the solution of (I V P).

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Example 2:- Solve Boundry value problem

$$\frac{d^2y}{dx^2} + y = 0 \quad , y(0) = 1, \quad y\left(\frac{\pi}{2}\right) = 2$$

Solution :- $(D^2+1)y = 0$

$$D^2 + 1 = 0$$

$$D = \pm i$$

So the general solution is $y = A\sin x + B\cos x$

Now we apply boundary condition

$$y(0) = A\sin(0) + B\cos(0)$$

$$1 = B$$

And $y\left(\frac{\pi}{2}\right) = A\sin(\pi/2) + B\cos(\pi/2)$

$$2 = A(1) + B(0)$$

$$A = 2$$

putting the value of A and B in general Solution

$$y = 2\sin x + \cos x$$

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I-V-P میں ایک کنڈیشن دی ہوتی ہے یا کوئی کنڈیشن کا Derivative دیا ہوتا ہے۔

جیسا کہ $y' = 2x \quad , y(1) = 4$

جبکہ B-V-P میں دو یا دو سے زیادہ کنڈیشن دی ہوتی ہیں۔ جیسا کہ

$y'' + y = 0 \quad , y(0) \text{ and } y(\pi/2) = 2$

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Eigen value Problems

If an initial value or boundary value problem contains a parameter (λ) in the D-E and non-trivial solution corresponding to certain values of λ can be found, the problem is called eigen value problem. The corresponding values of λ are called eigen values of the problem.

An eigen value problem is illustrated below.

Illustrative Example

Solve the problem, $$\frac{d^2u}{dx^2} + \lambda u = 0 \quad \text{——(i)}$$

where λ is parameter, and the boundary Conditions are $u(0) = 0$, $u(a) = 0$ ——(ii)

Solution :- The given problem is a boundary value problem involving a parameter λ. That is clear it is a eigen value problem. it is clear $u=0$ is a trivial Solution of the problem. But we look for a "non trivial solution" and therefor try to solve the given D-E Corresponding to different cases from λ being zero, +ve or -ve.

Case I :- when $\lambda = 0$.

From (i) $\frac{d^2u}{dx^2} = 0 \Rightarrow \frac{du}{dx} = A$

again integrating

$$u = Ax + B$$

where A and B are arbitrary Constant

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Now we apply the given boundary conditions to determine the unknown Constants.

First B.Cs $u(0) = A(0) + B$

$0 = B \Rightarrow \boxed{B = 0}$

2nd B.Condition

$u(a) = A(a) + B$

$0 = A + 0$

$\boxed{A = 0}$

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This solution satisfing both the condition is $u = 0$, which is trivial solution.

In other words, $\lambda = 0$ is not an eigen value of the give problem.

یاد رہے کہ λ کا ایسی ویلیو جو زیرو نہ ہو اس کو eigen value کہیں گے
اس لئے $\lambda = 0$ eigen ویلیو نہیں ہے

Case II :- when $\lambda < 0$ we may take $\lambda = -\mu^2$ and $\mu \neq 0$, there for the given D.E becomes

$$\frac{d^2u}{dx^2} - \mu^2 u = 0$$

$$(D^2 - \mu^2)u = 0 \quad , D = d/dx$$

The auxilairy equation is

$$m^2 - \mu^2 = 0$$

$$m = \pm \mu$$

So the general solution is $u = Ae^{\mu x} + Be^{-\mu x}$

The Boundary Condition $u(0) = 0$ gives

$$A + B = 0 \quad \text{---(iii)}$$

And 2nd Boundary Condition $u(a) = 0$ gives

$$Ae^{\mu a} + Be^{-\mu a} = 0 \quad \text{---(iv)}$$

Foom (iii) and (iv)

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$$\begin{bmatrix} 1 & 1 \\ e^{4a} & e^{-4a} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (10)$$

So $\begin{vmatrix} 1 & 1 \\ e^{4a} & e^{-4a} \end{vmatrix} = 0$

$$e^{-4a} - e^{4a} = 0$$

$$e^{-4a} = e^{4a}$$

$$-4a = 4a$$

$$2 4a = 0$$

Since "2" and "a" are non zero

There for $4 = 0$

But $4 \neq 0$ which is Contradiction. Hence $\lambda < 0$ is not eigen value

Case III :- when $\lambda > 0$

we may take $\lambda = 4^2$, $4 \neq 0$

There for the given D-E becomes

$$\frac{d^2u}{dx^2} + 4^2 u = 0$$

$$(D^2 + 4^2)u = 0$$

The auxiliary equation $m^2 + 4^2 = 0$

$$m^2 = -4^2 \Rightarrow m = \pm i4$$

So the general solution can be written as

$$u = A\cos 4x + B\sin 4x$$

Now we apply boundary Conditions

$$u(0) = A\cos 4(0) + B\sin 4(0)$$

$$0 = A + 0$$

$$\boxed{A = 0}$$

Now for 2nd B.C

$$u(a) = A\cos 4(a) + \sin 4(a)$$

$$0 = \sin 4(a)$$

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$$y(a) = \sin^{-1}(0)$$

$$y(a) = n\pi \qquad n \in \mathbb{Z}$$

$$y = y_n = \frac{n\pi}{a} \qquad n \in \mathbb{Z}$$

It follows that $\lambda_n = y_n^2 = \frac{n^2\pi^2}{a^2}$ are the eigen values. we note that non-trivial solutions exist only for these discrete values of λ which are the eigen values of the problem.

The corresponding solutions called eigen solutions are given by

$$u = u_n = C_n \sin \lambda_n x$$

where $\lambda_n = \frac{n^2\pi^2}{a^2}$

Orthogonal and Orthonormal Functions

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Definitions

(1) Functions $f(x)$ and $g(x)$ defined over the interval $a \le x \le b$ are said to be orthogonal if

$$\langle f, g \rangle = \int_a^b f(x) \cdot g(x)\, dx = 0$$

(2) - A set of functions $\{f_n(x)\}$, $n = 1, 2, 3 \ldots$ is said to be an orthogonal set if

$$\langle f_m(x) \cdot g_n(x) \rangle = \int_a^b f_m(x) \cdot g_n(x)\, dx = 0 \qquad m \neq n$$

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(3) A set of functions $\{f_n(x)\}$, $n = 1, 2, 3 \ldots$ is said to be an orthonormal set if

$$\langle f_n(x), g_n(x) \rangle = \int_a^b f_m(x) \cdot g_n(x) = 1, \text{ if } m = n$$

$$= 0 \text{ if } m \neq n$$

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S-L SYSTEMS

Consider Second-order D-E of the foorm

$$p_0(x)\frac{d^2u}{dx^2} + p_1(x)\frac{du}{dx} + p_2(x)u(x) = 0$$

where $p_0(x) \neq 0$

where

$p_0(x)$, $p_1(x)$, $p_2(x)$ are real and continous functions over $[a,b]$

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S-L Equation :-

The above Second order linear D-E can be written as in the form

$$\frac{d}{dx}\left\{P(x)\frac{du}{dx}\right\} + q(x)u(x) + \lambda w(x)u(x) = 0 \quad \text{---(1)}$$

which is called S-L equation.

To express Second-order D-E in S-L form first we write Second-order D-E.

$$p_0(x)\cdot\frac{d^2u}{dx^2} + p_1(x)\frac{du}{dx} + p_2(x)\cdot u(x) = 0$$

or $p_0(x)u'' + p_1(x)u' + p_2(x)u(x) = 0$

where $p_0(x) \neq 0$

Divided by $p_0(x)$

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$$u'' + \frac{p_1(x)}{p_0(x)} u' + \frac{p_2(x)}{p_0(x)} u = 0$$

multiply by $v(x)$

$$v \cdot u'' + v \frac{p_1(x)}{p_0(x)} u' + v \frac{p_2(x)}{p_0(x)} u = 0$$

$$\frac{d}{dx}(v \cdot u') + v \frac{p_2(x)}{p_0(x)} u = 0 \quad \text{——— (2)}$$

$$\text{So } v = \exp\left(\int_{x_0}^{x} \frac{p_1(x)}{p_0(x')} dx' \right)$$

Now Comparing (1) & (2)

$$p(x) = v(x) = \exp\left(\int_{x_0}^{x} \frac{p_1(x')}{p_0(x')} dx' \right)$$

$$q(x) = v(x) \frac{p_2(x)}{p_0(x)}$$

$$w(x) = v(x) \frac{p_2(x)}{p_0(x)}$$

Hence Equation (2) becomes in S-L form

$$\frac{d}{dx}\left\{ p(x) \frac{du}{dx} \right\} + q(x) u + \lambda w(x) u = 0$$

The above Second-order ordinary homogenous D-E is called S-L Equation if $p(x)$, $p'(x)$ $q(x)$, $w(x)$ are real and Continous over an interval $[a, b]$.

If we define an operator L as

$$L = \frac{d}{dx}\left\{ p(x) \frac{d}{dx} \right\} + q(x)$$

then the S-L - equation can be written as

$$L(u) + \lambda w(x) u = 0$$

Note L is linear operator:

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Regular S-L Equation :-

An S-L equation is said to be a regular S-L equation if $p(x)$ and $w(x)$ dont vanish in $[a,b]$

Example :- $u'' + \lambda u = 0$ —(1)

(i) can be written as

$$\frac{d}{dx}\left\{1 \cdot \frac{du}{dx}\right\} + 0 \cdot u(x) + \lambda \cdot 1 \cdot u(x) = 0$$

Comparing with S-L equation

$$\frac{d}{dx}\left\{p(x) \cdot \frac{du}{dx}\right\} + q(x) \cdot u(x) + \lambda\, w(x) \cdot u(x) = 0$$

$p(x) = 1$, $q(x) = 0$, $w(x) = 1$

Hence $p(x) > 0, w(x) > 0$ and represent Regular S-L equation.

Singular S-L Equation :-

An S-L equation is said to be a Singular S-L equation if $p(x)$ or $w(x)$ vanishes in interval $[a,b]$.

Example :- $\frac{d}{dx}\left\{(x^2-1)u'\right\} + \lambda u(x) = 0 \quad ; [-1,1]$

$$\frac{d}{dx}\left\{(x^2-1) \cdot \frac{du}{dx}\right\} + 0 \cdot u(x) + \lambda u(x) = 0$$

Comparing with general form of S-L Equation

$p(x) = x^2 - 1$ $\quad q(x) = 0$ $\quad w(x) = 1$

It is clear $p(x)$ is vanish at $x = \pm 1$

Hence, it represents a Singular S-L equation over a interval $[-1,1]$

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Periodic S-L equation :-

The S-L equation

$$\frac{d}{dx}\left\{p(x).\frac{du}{dx}\right\}+q(x)u+\lambda w(x)u(x)=0$$

is said to be periodic s-l equation if $p(x)$, $q(x)$, $w(x)$ are periodic functions of period "b-a".

i-e $p(a)=p(a+(b-a))=p(b)$

$q(a)=q(a+(b-a))=q(b)$

$w(a)=w(a+(b-a))=w(b)$

Example :- The Mathieu D-E

$$u''+(16\alpha\cos 2x)u+\lambda u=0$$

above D-E can be written as

$$\frac{d}{dx}\left[1.\frac{du}{dx}\right]+16\alpha\cos 2x\,u(x)+\lambda\cdot 1\cdot u(x)=0$$

Comparing with General S-L equation

$p(x)=1$, $q(x)=16\alpha\cos 2x$ $w(x)=1$

It represents periodic S-L equation.

Boundary Conditions associated with S-L Systems

Separated Boundary Conditions :-

The Boundary conditions

$$\alpha_1 u(a)+\alpha_2 u'(a)=0$$

$$\beta_1 u(b)+\beta_2 u'(b)=0$$

are called separated Boundary Conditions.

Periodic boundary Condition :-

If the separated boundary conditions are the form

$$u(a)=u(b),\quad u'(a)=u'(b)$$

Then they are called periodic boundary conditions.

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Dirichlet Boundary Condition :- If the separated Boundary conditions are the form

$$u(a) = c_1 \quad , u(b) = c_2$$

then they are called Dirichlet B.Cs.

Neumann Boundary Conditions :- Finally if the separated B.Cs are the form

$$u'(a) = c_1' \quad , u'(b) = c_2'$$ then they are called Neumann B.Cs

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REGULAR AND PERIODIC S-L SYSTEMS

① Regular S-L system :- The regular S-L equ

$$\frac{d}{dx}\left\{P(x)\cdot\frac{du}{dx}\right\} + V(x)u(x) + \lambda w(x)u(x) = 0$$

is said to form a regular S-L system over the interval $[a,b]$, if it satisfied the following Conditions

$$\alpha_1 u(a) + \alpha_2 u'(a) = 0$$

A/2011 $$\beta_1 u(b) + \beta_2 u'(b) = 0$$

② Periodic S-L system :-

The periodic S-L equation along with end conditions

$$u(a) = u(b) \quad , u'(a) = u'(b)$$

is called periodic S-L system.

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Eigen values And Eigen Solution of an S-L System

Definition :- A non zero $u(x)$ of an S-L system (regular or periodic) is said to be an eigensolution or eigen function corresponding to the value of parameter λ. And the value of λ is called an eigen value of the differential equation.

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Question :- Find the eigen values and eigen functions for S-L system

$$y''(x) + \lambda y(x) = 0$$

$$y(0) = 0 \quad , \quad y(\pi) = 0$$

for $\lambda < 0$ and $\lambda > 0$.

Solution :- [Case I] For $\lambda = 0$

$$y''(x) + \lambda y(x) = 0$$

$$y''(x) = 0$$

$$y'(x) = A \Rightarrow y(x) = Ax + B$$

Now we apply B.C's:

$$y(0) = 0 \Rightarrow A(0) + B = 0$$

$$\boxed{B = 0}$$

and $y(\pi) = 0 \Rightarrow A(\pi) + B = 0$

$$A\pi = 0 \quad \text{since } B = 0$$

$$\pi \neq 0 \quad \text{So} \quad A = 0$$

Hence $\lambda = 0$ is not eigen value.

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Case II :- $\lambda < 0$

Take $\lambda = -\mu^2$; $\mu > 0$

So the given equation becomes

$$y''(x) - \mu^2 y(x) = 0$$

$$(D^2 - \mu^2) y = 0$$

For non-trivial Solution $D^2 - \mu^2 = 0$

$$D = \pm \mu$$

So the general solution is

$$y(x) = Ae^{\mu x} + Be^{-\mu x}$$

Now we apply B-C's (i) $y(0) = 0$

$$Ae^0 + Be^0 = 0$$

$$A + B = 0 \quad \text{---} (1)$$

And second B-Condition $y(\pi) = 0$

$$Ae^{\mu\pi} + Be^{-\mu\pi} = 0 \quad \text{---} (2)$$

From equation (1) and (2)

$$\begin{vmatrix} 1 & 1 \\ e^{\mu\pi} & e^{-\mu\pi} \end{vmatrix} \begin{vmatrix} A \\ B \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

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For non-trivial Solution $A \neq 0$, $B \neq 0$

So

$$\begin{vmatrix} 1 & 1 \\ e^{\mu\pi} & e^{-\mu\pi} \end{vmatrix} = 0$$

$$e^{-\mu\pi} - e^{\mu\pi} = 0$$

$$e^{\mu\pi} = e^{-\mu\pi}$$

$$\mu\pi = -\mu\pi$$

$$\mu = -\mu \qquad \text{But } \mu > 0$$

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which is Contradiction.

Hence $\lambda < 0$ is not eigen value.

Case III :- $\lambda > 0$

Take $\lambda = \mu^2$, $\mu > 0$

So the given D-E becomes

$$y''(x) + \mu^2 y(x) = 0$$

The chasteristic equation is

$$(D^2 + \mu^2) y = 0$$

For non-trivial solution $y \neq 0$

So $D^2 + \mu^2 = 0$

$$D = \pm i\mu$$

The General Solution is

$$y(x) = A\cos\mu x + B\sin\mu x$$

Now applying B.C's (i) $y(0) = 0$

$$A\cos(0) + B\sin(0) = 0$$

$$A\cos(0) = 0$$

$$\boxed{A = 0}$$

2nd B.Condition $y(\pi) = 0$

$$A\cos\mu\pi + B\sin\pi\mu = 0$$

$\because$ Since $A = 0 \Rightarrow B\sin\pi\mu = 0$

For non-trivial Solution $B \neq 0$

$$\sin\pi\mu = 0$$

$$\pi\mu = \sin^{-1}(0)$$

$$\pi\mu = n\pi \qquad n = 0, \pm 1, \pm 2$$

$$\mu = n$$

$$\sqrt{\lambda_n} = n$$

Hence $\lambda = \lambda_n = n^2$; $n \in \mathbb{Z}$ are eigen values of λ for which non trivial solution can be found. Hence there Corresponding eigen function is $y_n(x) = B_n \sin\sqrt{\lambda_n}\, x$

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Q :- Determine the eigen values of the system $u'' + \lambda u = 0$

with $u(0) = u(\pi)$; $u'(0) = u'(\pi)$

Solution :-

Given that $u'' + \lambda u = 0$ —(i)

we will discuss three cases

Case I :- $\lambda = 0$

Equation (i) becomes $u'' = 0$

$u' = A$

$u = Ax + B$ —(ii)

Now we apply B.C's $u(0) = u(\pi)$

$A(0) + B = A\pi + B$

$0 = A\pi$

Since $\pi \neq 0$, So $A = 0$

Now Equation (ii) becomes $u = B$

$u' = 0$

So second B.C does not help for finding the value of B

So $u(x) = B \neq 0$

Hence for $\lambda = 0$ is an eigen value of given D-E where B is arbitrary Constant.

Case II :- $\lambda < 0$

Let $\lambda = -\mu^2$, $\mu > 0$

So (i) becomes $u'' - \mu^2 u = 0$

The characteristic equation is

$(D^2 - \mu^2) u = 0$

For non-trivial Solution $u \neq 0$

$\therefore$ So $D^2 - \mu^2 = 0 \Rightarrow D = \pm \mu$

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The general solution is $u(x) = Ae^{4x} + Be^{-4x}$

Now we applying B.C's $u(0) = u(\pi)$

$$Ae^{0} + Be^{0} = Ae^{4\pi} + Be^{-4\pi}$$

$$A + B - Ae^{4\pi} - Be^{-4\pi} = 0$$

$$A(1 - e^{4\pi}) + B(1 - e^{-4\pi}) = 0 \quad \text{---(iii)}$$

Now 2nd B.Condition

$$u'(0) = u'(\pi)$$

$$u'(x) = 4Ae^{4x} - 4Be^{-4x}$$

$$4[Ae^{0} - Be^{0}] = 4[Ae^{4\pi} - Be^{-4\pi}]$$

$$A - B - Ae^{4\pi} + Be^{-4\pi} = 0$$

$$A[1 - e^{4\pi}] + B[-1 + e^{-4\pi}] = 0 \quad \text{---(iv)}$$

From equation (iii) and (iv)

$$\begin{vmatrix} 1 - e^{4\pi} & 1 - e^{-4\pi} \\ 1 - e^{4\pi} & -1 + e^{-4\pi} \end{vmatrix} \begin{vmatrix} A \\ B \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

For non-trivial Solution $A \neq 0, B \neq 0$

$$\begin{vmatrix} 1 - e^{4\pi} & 1 - e^{-4\pi} \\ 1 - e^{4\pi} & -1 + e^{-4\pi} \end{vmatrix} = 0$$

$$(1 - e^{4\pi})(-1 + e^{-4\pi}) - (1 - e^{4\pi})(1 - e^{-4\pi}) = 0$$

$$(1 - e^{4\pi})[-1 + e^{-4\pi} - 1 + e^{-4\pi}] = 0$$

$$(1 - e^{4\pi})(-2 + 2e^{-4\pi}) = 0$$

$$2(1 - e^{4\pi})(-1 + e^{-4\pi}) = 0$$

$$(1 - e^{4\pi})(-1 + e^{-4\pi}) = 0 \quad ; 2 \neq 0$$

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$$-1+e^{-4\pi}+e^{4\pi}-e^{4\pi}\cdot e^{-4\pi}=0$$

$$-1+e^{-4\pi}+e^{4\pi}-1=0$$

$$-2+e^{-4\pi}+e^{4\pi}=0$$

$$-2+\frac{1}{e^{4\pi}}+e^{4\pi}=0$$

$$-2e^{4\pi}+1+e^{24\pi}=0$$

$$e^{24\pi}-2e^{4\pi}+1=0 \quad \text{—(v)}$$

put $e^{4\pi}=t$

(v) becomes $t^2-2t+1=0$

$a=1$, $b=-2$, $c=1$

$$t=\frac{2\pm\sqrt{4-4}}{2}$$

$t=1$

$e^{4\pi}=1$

$e^{4\pi}=e^0$ $\therefore \boxed{e^0=1}$

$4\pi=0$

Since $\pi\neq 0$ $4=0$ But $4>0$ which is contradiction Hence $\lambda<0$ is not eigen value.

Case III :- $\lambda>0$

Let $\lambda=4^2$ where $4>0$

So equation (i) becomes $u''+4u=0$

The characteristic equ $(D^2+4^2)u=0$

for non trivial Solution $u\neq 0$

$D^2+4^2=0$

$D=\pm i4$

The general Solution is

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$$u(x) = A\cos\mu x + B\sin\mu x \quad \text{—(vi)}$$

Now we applying B.C's $u(0) = u(\pi)$

$$A\cos(0) + B\sin(0) = A\cos\pi\mu + B\sin\pi\mu$$

$$A - A\cos\mu\pi - B\sin\pi\mu = 0$$

$$A[1 - \cos\mu\pi] - B[\sin\pi\mu] = 0 \quad \text{—(vii)}$$

Now apply 2nd B.C

$$\therefore u(x) = A\cos\mu x + B\sin\mu x$$

$$u'(x) = -\mu A\sin\mu x + \mu B\cos\mu x$$

$$u'(x) = \mu[-A\sin\mu x + B\cos\mu x]$$

Since $u'(0) = u'(\pi)$

$$\mu[-A\sin 0 + B\cos 0] = \mu[-A\sin\mu\pi + B\cos\mu\pi]$$

$$B = -A\sin\mu\pi + B\cos\mu\pi$$

$$B + A\sin\mu\pi - B\cos\mu\pi = 0 \quad \text{—(viii)}$$

$$A\sin\mu\pi + B(1 - \cos\mu\pi) = 0$$

From (vii) and (viii)

$$\begin{vmatrix} 1-\cos\mu\pi & -\sin\mu\pi \\ \sin\mu\pi & 1-\cos\mu\pi \end{vmatrix} \begin{matrix} A \\ B \end{matrix} = \begin{matrix} 0 \\ 0 \end{matrix}$$

For non-trivial solution $A \neq 0,\ B \neq 0$

$$\begin{vmatrix} (1-\cos\mu\pi) & -\sin\mu\pi \\ \sin\mu\pi & (1-\cos\mu\pi) \end{vmatrix} = 0$$

$$(1-\cos\mu\pi)(1-\cos\mu\pi) - (-\sin\mu\pi)(\sin\mu\pi) = 0$$

$$1 + \cos^2\mu\pi - 2\cos\mu\pi + \sin^2\pi = 0$$

$$1 - 2\cos\mu\pi + 1 = 0$$

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$$2 - 2\cos 4\pi = 0$$

$$2\cos 4\pi = 2$$

$$\cos 4\pi = 1$$

$$4\pi = \cos^{-1}(1)$$

$$4\pi = 2n\pi \quad ; n \in \mathbb{Z}$$

Since $4 \neq 0$ as determinant is non-zero

therefor $A = 0$, $B = 0$

Hence $\lambda > 0$ is not an eigen value

★ ★

A/2015, A/2013

Question :- Determine the eigen value and eigen function for the following S-L problem.

$$u''(x) + \lambda u(x) = 0$$

$$u(0) = 0 \quad , \quad u'(\pi/2) = 0$$

Solution :-

Given that $u''(x) + \lambda u(x) = 0$

The charteristic equation is

$$(D^2 + \lambda) u = 0$$

For non - trivial Solution $x \neq 0$

$$D^2 + \lambda = 0$$

$$D = \pm i\sqrt{\lambda}$$

So the general Solution is

$$u = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x \quad \text{—(i)}$$

Now we apply B.C,s $u(0) = 0$

$$A\cos(0) + B\sin(0) = 0$$

$$A(1) + 0 = 0$$

$$\boxed{A = 0}$$

Differentiate (i) for applying Second B.C

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$$u'(x) = \sqrt{\lambda}\left[-A\sin\sqrt{\lambda}\,x + B\cos\sqrt{\lambda}\,x\right]$$

As $u'(\pi/2) = 0$

$$\sqrt{\lambda}\left[-A\sin\sqrt{\lambda}\left(\frac{\pi}{2}\right) + B\cos\sqrt{\lambda}\,(\pi/2)\right] = 0$$

For non-trivial Solution $\sqrt{\lambda} \neq 0$

$$-A\sin\sqrt{\lambda}\,(\pi/2) + B\cos\sqrt{\lambda}\,(\pi/2) = 0$$

Since $\boxed{A=0}$ Therefor $B\cos\sqrt{\lambda}\,(\pi/2) = 0$

For non-trivial Solution $B \neq 0$

So $\cos\sqrt{\lambda}\,(\pi/2) = 0$

$$\sqrt{\lambda}\,(\pi/2) = \cos^{-1}(0)$$

$$\sqrt{\lambda}\,(\pi/2) = (2n+1)\frac{\pi}{2} \quad n \in Z$$

$$\sqrt{\lambda} = 2n+1$$

$$\lambda_n = (2n+1)^2$$

are eigen values and there corresponding eigen functions are

$$u_n(x) = B_n(2n+1)x$$

A/2011

Question :- Find the eigen values and eigen functions of S-L problem

$$y''(x) + \lambda y(x) = 0$$

$$y'(0) = 0 \quad , y(\pi) = 0$$

Solution :- Given $y''(x) + \lambda y(x) = 0$

The charteristic equation is

$$(D^2 + \lambda)y = 0$$

For non-trivial Solution $y \neq 0$

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So $D^2 + \lambda = 0 \Rightarrow D = \pm i\sqrt{\lambda}$

The general solution is

$$y(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x \quad \text{---(i)}$$

$$y'(x) = \sqrt{\lambda}[-A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x]$$

Now we applying B.C's

$$y'(0) = 0$$

$$\sqrt{\lambda}[-A\sin(0) + B\cos(0)] = 0$$

For non-trivial Solution $\sqrt{\lambda} \neq 0$

So $-A(0) + B(1) = 0$

$$\boxed{B = 0}$$

Now Second B.Condition $y(\pi) = 0$

$$A\cos\sqrt{\lambda}(\pi) + B\sin\sqrt{\lambda}(\pi) = 0$$

Since $B = 0$

Therefor $A\cos\sqrt{\lambda}(\pi) = 0$

For non-trivial Solution $A \neq 0$

$$\cos\sqrt{\lambda}(\pi) = 0$$

$$\sqrt{\lambda}\,\pi = \cos^{-1}(0)$$

$$\sqrt{\lambda}\,\pi = (2n+1)\frac{\pi}{2} \quad n \in \mathbb{Z}$$

$$\sqrt{\lambda_n} = n + \frac{1}{2}$$

are eigen values and their corresponding Eigen functions are

$$y_n(x) = A_n \cos\left(n + \frac{1}{2}\right)x$$

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A/2016 {3(b)} , A/2014 {3(b)}

Question :- Show that $\lambda_n = \left(\frac{n\pi}{\ln b}\right)^2$ and $u_n = B_n \sin(\lambda_n \ln x)$ are eigen values and eigen functions of the S-L system.

$$(xu')' + \lambda \frac{1}{x} u = 0 \quad ; \quad 0 < x < b$$

$$u(1) = 0 \quad , \quad u(b) = 0$$

پیپر سوال میں $u(0)=0$ دیا ہوا ہے جو کہ غلط ہے اس سے سوال حل نہیں ہو گا بلکہ $u(1)=0$ ہونا چاہیے جو کہ سوال سے لیا گیا ہے۔

Solution :-

Given $(xu')' + \lambda \cdot \frac{1}{x} u = 0$

$$xu'' + u' + \lambda \frac{u}{x} = 0$$

multiply by "x" $x^2u'' + xu' + \lambda u = 0$ ——— (i)

It is Cauchy-Euler D-E

Solve this D-E put $t = \ln x$

$$\frac{dt}{dx} = \frac{1}{x}$$

And $\frac{du}{dx} = \frac{du}{dt} \times \frac{dt}{dx}$

$$\frac{du}{dx} = \frac{du}{dt} \cdot \frac{1}{x} \quad \text{——— (ii)}$$

$D \equiv \Delta \cdot \frac{1}{x}$ where $\frac{du}{dx} = D$ & $\frac{du}{dt} = \Delta$

$xD = \Delta$ ——— (iii)

Now equation (ii) again Diff w.r.t "x"

$$\frac{d^2u}{dx^2} = \frac{du}{dt}\left[-\frac{1}{x^2}\right] + \left[\frac{1}{x}\right] \cdot \frac{d^2u}{dt^2} \cdot \frac{dt}{dx}$$

$$\frac{d^2u}{dx^2} = -\frac{1}{x^2}\frac{du}{dt} + \frac{1}{x} \cdot \frac{d^2u}{dt^2} \cdot \left(\frac{1}{x}\right)$$

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$$\frac{d^2u}{dx^2} = \frac{1}{x^2}\left[-\frac{du}{dt} + \frac{d^2u}{dt^2}\right]$$

$$D^2 = \frac{1}{x^2}\left[-\Delta + \Delta^2\right]$$

$$x^2D^2 = \Delta^2 - \Delta \quad \text{---(iv)}$$

Now take equation (1)

$$x^2u'' + xu' + \lambda u = 0$$

$$(x^2D^2 + xD + \lambda)u = 0$$

For non-trivial solution $u \neq 0$

$$x^2D^2 + xD + \lambda = 0$$

$$\Delta^2 - \Delta + \Delta + \lambda = 0$$

$$\Delta^2 + \lambda = 0$$

$$\Delta = \pm i\sqrt{\lambda}$$

The general solution is

$$u(t) = A\cos\sqrt{\lambda}\,t + B\sin\sqrt{\lambda}\,t$$

But $t = \ln x$

$$u(x) = A\cos\sqrt{\lambda}(\ln x) + B\sin\sqrt{\lambda}(\ln x)$$

Now apply B.C's $u(1) = 0$

$$A\cos\sqrt{\lambda}(\ln 1) + B\sin\sqrt{\lambda}(\ln 1) = 0$$

$$A\cos(0) + B\sin(0) = 0$$

$$\boxed{A = 0}$$

And 2nd Boundary condition is $u(b) = 0$

$$A\cos\sqrt{\lambda}\{\ln(b)\} + B\sin\sqrt{\lambda}(\ln b) = 0 \quad \text{---(v)}$$

Since $A = 0$

Hence for (v) becomes

$$B\sin\sqrt{\lambda}(\ln b) = 0$$

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For non-trivial Solution $B \neq 0$

$$\sin \sqrt{\lambda}(\ell n\, b) = 0$$

$$\sqrt{\lambda}(\ell n\, b) = \sin^{-1}(0)$$

$$\sqrt{\lambda}(\ell n\, b) = n\pi \quad ; n \in \mathbb{Z}$$

$$\sqrt{\lambda} = \frac{n\pi}{\ell n\, b}$$

$$\lambda_n = \left(n\pi / \ell n\, b\right)^2$$

Hence the eigen values are $\lambda_n = \left(\frac{n\pi}{\ell n\, b}\right)^2$ and eigen functions are

$$u_n(x) = B_n \sin \sqrt{\lambda_n}(\ell n\, x)$$

Question :- Determine the eigen values of the system $u'' + \lambda u = 0$ with B.C's $u(0) = u(\pi)$

$$u'(0) = 2u'(\pi)$$

Solution :-

Given that $u'' + \lambda u = 0$

The charteristic equation is

$$(D^2 + \lambda)u = 0$$

For non-trivial Solution $u \neq 0$

$$D^2 + \lambda = 0$$

$$D = \pm i\sqrt{\lambda}$$

The general Solution is

$$u(x) = A\cos\sqrt{\lambda}\, x + B \sin\sqrt{\lambda}\, x \quad \text{---} (1)$$

Now we applying Bamday Conditions

$$u(0) = u(\pi)$$

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$A\cos(0) + B\sin(0) = A\cos\sqrt{\lambda}(\pi) + B\sin\sqrt{\lambda}(\pi)$

$A + 0 = A\cos\sqrt{\lambda}(\pi) + B\sin\sqrt{\lambda}(\pi)$

$0 = -A + \cos\sqrt{\lambda}(\pi) + B\sin\sqrt{\lambda}(\pi)$

$A(\cos\sqrt{\lambda} - 1) + B\sin\sqrt{\lambda}(\pi) = 0$ ——— (2)

Now Differentiate equation (1) for applying 2nd B.Cs

$u'(x) = A(-\sin\sqrt{\lambda}x)\cdot\sqrt{\lambda} + B(\cos\sqrt{\lambda}x)\cdot\sqrt{\lambda}$

$u'(x) = \sqrt{\lambda}[-A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x]$

As 2nd B.C $u'(0) = 2u'(\pi)$

$\sqrt{\lambda}[-A\sin(0) + B\cos(0)] = 2[\sqrt{\lambda}\{-A\sin\sqrt{\lambda}\pi + B\cos\sqrt{\lambda}\pi]$

$0 + B = 2[-A\sin\sqrt{\lambda}\pi + B\cos\sqrt{\lambda}\pi]$

$2A\sin\sqrt{\lambda}\pi - 2B\cos\sqrt{\lambda}\pi + B = 0$

$2A\sin\sqrt{\lambda}\pi + B[1 - 2\cos\sqrt{\lambda}\pi] = 0$ ——— (3)

From (2) & (3) it can be written as

$$\begin{vmatrix} \cos\sqrt{\lambda}-1 & \sin\sqrt{\lambda}\pi \\ 2\sin\sqrt{\lambda}\pi & 1-2\cos\sqrt{\lambda}\pi \end{vmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For non-trivial solution $A \neq 0$ & $B \neq 0$

$$\begin{vmatrix} \cos\sqrt{\lambda}-1 & \sin\sqrt{\lambda}\pi \\ 2\sin\sqrt{\lambda}\pi & 1-2\cos\sqrt{\lambda}\pi \end{vmatrix} = 0$$

$(\cos\sqrt{\lambda}\pi - 1)(1 - 2\cos\sqrt{\lambda}\pi) - (\sin\sqrt{\lambda}\pi)(2\sin\sqrt{\lambda}\pi) = 0$

$\cos\sqrt{\lambda}\pi - 2\cos^2\sqrt{\lambda}\pi - 1 + 2\cos\sqrt{\lambda}\pi - 2\sin^2\sqrt{\lambda}\pi = 0$

$3\cos\sqrt{\lambda}\pi - 2[\cos^2\sqrt{\lambda}\pi + \sin^2\sqrt{\lambda}\pi] - 1 = 0$

$3\cos\sqrt{\lambda}\pi - 2(1) - 1 = 0$

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$3\cos\sqrt{\lambda}\pi - 3 = 0$

$3\cos\sqrt{\lambda}\pi = 3$

$\cos\sqrt{\lambda}\pi = 1$

$\sqrt{\lambda}\pi = \cos^{-1}(1)$

$\sqrt{\lambda}\pi = 2n\pi$

$\sqrt{\lambda} = 2n$

$\Rightarrow \lambda_n = 4n^2$; $n = 0, 1, 2, \ldots$

are the eigen values and their corresponding eigen functions are

$$u_n(x) = A_n \cos 2nx + B_n \sin 2nx$$

ORTOGONALITY Of EIGEN FUNCTION Of S-L SYSTEM

Definition:- Functions $\phi(x)$ and $\psi(x)$ defined over the interval $[a, b]$ are said to be orthogonal w.r.t a weight function $w(x)$ if

$$\int_a^b \phi(x) \cdot \psi(x) \cdot w(x)\, dx = 0$$

Now we find the orthogonality solution of S-L systems (regular and Periodic):

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A/2012 (33)

Orthogonality of solution of Regular S-L system

Theorem :- If $u_m(x)$ and $u_n(x)$ are solutions of a regular S-L system corresponding to distinct eigen values λ_m and λ_n of parameter λ, then $u_m(x)$ and $u_n(x)$ are orthogonal w.r.t the weight function, $w(x)$, i-e

$$\int_a^b u_m(x) \cdot u_n(x) \cdot w(x) = 0 \quad ; \quad m \neq n$$

OR

Show that the eigen functions corresponding to distinct eigen values of regular S-L system are orthogonal

Proof :- Consider the regular S-L system

$$\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u(x) + \lambda w(x)u(x) = 0$$

It can be written as

$$(p(x)u'(x))' + q(x)u(x) + \lambda w(x)u(x) = 0 \quad \text{---(1)}$$

with Boundary Conditions

$$\left.\begin{aligned} \alpha_1 u(a) + \alpha_2 u'(a) &= 0 \\ \beta_1 u(b) + \beta_2 u'(b) &= 0 \end{aligned}\right\} \quad \text{---(2)}$$

Since $u_m(x)$ is the solution of regular S-L system, So it must satisfy — (1)

$$\left(p(x)u_m'(x)\right)' + q(x)u_m(x) + \lambda_m w(x)u_m(x) = 0 \quad \text{---(3)}$$

along with Boundary Conditions

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$$\left.\begin{aligned} \alpha_1 U_m(a) + \alpha_2 U_m'(a) &= 0 \\ \beta_1 U_m(b) + \beta_2 U_m'(b) &= 0 \end{aligned}\right\} \quad (4)$$

Similarly, As $U_n(x)$ is the solution of regular S-L system, So it must satisfy eqn (1)

$$\left(P(x)\,U_n'(x)\right)' + V(x)\,U_n(x) + \lambda_n\,\omega(x)\,U_n(x) = 0 \quad (5)$$

$$\left.\begin{aligned} \alpha_1 U_n(a) + \alpha_2 U_n'(a) &= 0 \\ \beta_1 U_n(b) + \beta_2 U_n'(b) &= 0 \end{aligned}\right\} \quad (6)$$

Multiply (3) by $U_n(x)$ and (5) by $U_m(x)$ and subtract

$$\left(P(x)\,U_m'(x)\right)' U_n(x) + V(x)\,U_m(x)\,U_n(x) + \lambda_m\,\omega(x)\,U_m(x)\,U_n(x) = 0$$

$$-\left(P(x)\,U_n'(x)\right)' U_m(x) + V(x)\,U_n(x)\,U_m(x) + \lambda_n\,\omega(x)\,U_n(x)\,U_m(x) = 0$$

$$\left(P(x)\,U_m'(x)\right)' U_n(x) - \left(P(x)\,U_n'(x)\right)' U_m(x) + \lambda_m\,\omega(x)\,U_m(x)\,U_n(x) - \lambda_n\,\omega(x)\,U_m(x)\,U_n(x) = 0$$

$$\left(P(x)\,U_m'(x)\right)' U_n(x) - \left(P(x)\,U_n'(x)\right)' U_m(x) + (\lambda_m - \lambda_n)\,\omega(x)\,U_m(x)\,U_n(x) = 0$$

$$\left(P(x)\,U_m'(x)\right)' U_n(x) - \left(P(x)\,U_n'(x)\right)' U_m(x) = (\lambda_n - \lambda_m)\,\omega(x)\,U_m(x)\,U_n(x) = 0$$

Integrating both side w.r.t "x" from a to b

$$\int_a^b \left(P(x)\,U_m'(x)\right)' U_n(x)\,dx - \int_a^b \left(P(x)\,U_n'(x)\right)' U_m(x)\,dx$$

$$= (\lambda_n - \lambda_m) \int_a^b \omega(x)\,U_n(x)\,U_m(x)\,dx$$

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$$\Rightarrow \left[U_n(x)\, p(x)\, U_m'(x) \right|_a^b - \int_a^b p(x)\, U_m'(x)\, U_n'(x)\, dx$$

$$- \left[U_m(x)\, p(x)\, U_n'(x) \right|_a^b + \int_a^b p(x)\, U_m'(x)\, U_n'(x) \cdot dx$$

$$= (\lambda_n - \lambda_m) \int_a^b w(x)\, U_n(x)\, U_m(x)\, dx$$

$$U_n(b)\, p(b)\, U_m'(b) - U_n(a)\, p(a)\, U_m'(a) - U_m(b)\, p(b)\, U_n'(b)$$

$$+ U_m(a)\, p(a)\, U_n'(a) = (\lambda_n - \lambda_m) \int_a^b w(x)\, U_m(x)\, U_n(x)\, dx \quad (7)$$

From (2) and (4)

$$U_n(b) = -\frac{\beta_2}{\beta_1} U_n'(b) \quad ; \quad U_n(a) = -\frac{\alpha_2}{\alpha_1} U_n'(a)$$

$$U_m(a) = -\frac{\alpha_2}{\alpha_1} U_m'(a) \quad ; \quad U_m(b) = -\frac{\beta_2}{\beta_1} U_m'(b)$$

putting these values in (7)

$$-\frac{\beta_2}{\beta_1} U_n'(b)\, p(b)\, U_m'(b) + \frac{\alpha_2}{\alpha_1} U_n'(a)\, p(a)\, U_m'(a)$$

$$+ \frac{\beta_2}{\beta_1} U_m'(b)\, p(b)\, U_n'(b) - \frac{\alpha_2}{\alpha_1} U_m'(a)\, p(a)\, U_n'(a)$$

$$= (\lambda_n - \lambda_m) \int_a^b w(x)\, U_m(x)\, U_n(x)\, dx$$

$$(\lambda_n - \lambda_m) \int_a^b w(x)\, U_m(x) \cdot U_n(x) \cdot dx = 0$$

As $(\lambda_n - \lambda_m) \neq 0$ Because eigen values distinct

so $$\int_a^b w(x)\, U_n(x)\, U_m(x) \cdot dx = 0$$

Hence, the eigen solutions are orthogonal.

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Orthogonality of Eigen Solution of Periodic S-L System

Theorem :- Poove that eigen solutions corresponding to distinct eigen values of Periodic S-L system are orthogonal w.r.t weight function $w(x)$.

Proof :- Consider the periodic S-L equation

$$(p(x)u'(x))' + q(x)u(x) + \lambda w(x)u(x) = 0 \quad \text{---(1)}$$

with B.C's

$$\left.\begin{aligned} u(a) &= u(b) \\ u'(a) &= u'(b) \end{aligned}\right\} \quad \text{---(2)}$$

Since $u_m(x)$ is the solution of the periodic S-L system, so it must satisfy (1)

$$(p(x)u_m'(x))' + q(x)u_m(x) + \lambda_m w(x)u_m(x) = 0 \quad \text{---(3)}$$

along with B.C's

$$\left.\begin{aligned} u_m(a) &= u_m(b) \\ u_m'(a) &= u_m'(b) \end{aligned}\right\} \quad \text{---(4)}$$

Similarly, As $u_n(x)$ is the solution of the periodic S-L system, so it must satisfy (1)

$$(p(x)u_n'(x))' + q(x)u_n(x) + \lambda_n w(x)u_n(x) = 0 \quad \text{---(5)}$$

along with B.C's

$$\left.\begin{aligned} u_n(a) &= u_n(b) \\ u_n'(a) &= u_n'(b) \end{aligned}\right\} \quad \text{---(6)}$$

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Multiply Equation ③ by $U_n(x)$ and ⑤ by $U_m(x)$ and then subtracting.

$$\left(p(x)U_m'(x)\right)'U_n(x) + q(x)U_m(x)U_n(x) + \lambda_m w(x)U_m(x)U_n(x) = 0$$

$$\left(p(x)U_n'(x)\right)'U_m(x) + q(x)U_n(x)U_m(x) + \lambda_n w(x)U_n(x)U_m(x) = 0$$

$$\left(p(x)U_m'(x)\right)'U_n(x) - \left(p(x)U_n'(x)\right)'U_m(x)$$

$$+ \lambda_m w(x)U_m(x)U_n(x) - \lambda_n w(x)U_n(x)U_m(x) = 0$$

$$\left(p(x)U_m'(x)\right)'U_n(x) - \left(p(x)U_n'(x)\right)'U_m(x) = (\lambda_n - \lambda_m)w(x)U_nU_m$$

integrating on both side w.r.t x from a to b

$$\int_a^b \left(p(x)U_m'(x)\right)'U_n(x) - \int_a^b \left(p(x)U_n'(x)\right)'U_m(x) = (\lambda_n - \lambda_m)\int_a^b w(x)U_nU_m$$

$$\left[U_n(x)\,p(x)U_m'(x)\right]_a^b - \int_a^b p(x)U_m'(x)U_n'(x) - \left[U_m(x)\,p(x)U_n(x)\right]_a^b$$

$$+ \int_a^b p(x)U_m'(x)U_n'(x) = (\lambda_n - \lambda_m)\int_a^b w(x)U_n(x)U_m(x)\,dx$$

$$U_n(b)\,p(b)U_m'(b) - U_n(a)\,p(a)U_m'(a)$$

$$- U_m(b)\,p(b)U_n(b) + U_m(a)\,p(a)U_n(a) = (\lambda_n - \lambda_m)\int_a^b w(x)U_nU_m$$

using the periodic - end conditions

$$U_n(b) = U_n(a)\ ;\ U_m'(b) = U_m'(a)$$

$$U_m(b) = U_m(a)\ ;\ U_n'(b) = U_n'(a)$$

and $p(a) = p(b)$

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$\Rightarrow U_n(a)\,p(a)\,U_m'(a) - U_n(a)\,p(a)\,U_m'(a) - U_m(a)\,p(a)\,U_n'(a) + U_m(a)\,p(a)\,U_n'(a) = (\lambda_n - \lambda_m)\int_a^b w(x)\,U_n(x)\,U_m$

$$(\lambda_n - \lambda_m)\int_a^b w(x)\,U_n(x)\,U_m(x)\,dx = 0$$

As $(\lambda_n - \lambda_m) \neq 0$ because eigen values are distinct.

Therefore $\int_a^b w(x)\,U_n(x)\cdot U_m(x)\,dx = 0$

This shows eigen solutions are orthogonal.

★ ★

A/2016

Question:- If $u(x)$ and $v(x)$ are periodic solution of the Mathieu equation with period π having distinct eigen values, Then prove that the solutions are orthogonal.

Solution:- The Mathieu equation with period π is

$$y'' + \lambda y + 16\alpha(\cos 2x)y = 0 \quad \text{—(1)}$$

with end point conditions

$$y(0) = y(\pi) \quad , y'(0) = y'(\pi)$$

Now u and v are solutions of (1) corresponding to distinct eigen values λ_1 & λ_2

Therefore $u'' + \lambda_1 u + 16\alpha(\cos 2x)u = 0$ —(2)

with B.C's

$$u(0) = u(\pi) \; ; \; u'(0) = u'(\pi)$$

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Similarly for "v", we have

$$v'' + \lambda_2 v + 16d(\cos 2x)v = 0 \quad \text{—③}$$

with B.C's $v(0) = v(\pi)$; $v'(0) = v'(\pi)$ —④

Multiply equation ② by "v" and ③ by u and subtract

$$u''v + \lambda_1 uv + 16d(\cos 2x)uv - v''u - \lambda_2 vu - 16d(\cos 2x)uv = 0$$

$$u''v - v''u + (\lambda_1 - \lambda_2)uv = 0$$

$$(\lambda_1 - \lambda_2)uv = v''u - u''v \quad \text{—⑤}$$

Taking integral of ⑤ from 0 to π, we get

$$(\lambda_1 - \lambda_2)\int_0^\pi u(x)v(x)\,dx = \int_0^\pi (v''u - u''v)\,dx$$

$$(\lambda_1 - \lambda_2)\int_0^\pi u(x)v(x)\,dx = \int_0^\pi (v''u + u'v' - u'v' - u''v)\,dx$$

$$= \int_0^\pi [(v''u + u'v') - (u'v' + u''v)]\,dx$$

$$= \int_0^\pi \left[\frac{d}{dx}(u\cdot v') - \frac{d}{dx}(u'\cdot v)\right]dx$$

$$= \int_0^\pi \frac{d}{dx}[u\cdot v' - u'v]\,dx$$

$$= u(x)v'(x) - u'(x)v(x)\Big]_0^\pi$$

$$= u(\pi)v'(\pi) - u'(\pi)v(\pi) - u(0)v'(0) + u'(0)v(0)$$

$$= u(\pi)v'(\pi) - u'(\pi)v(\pi) - u(\pi)v(\pi) + u'(\pi)v(\pi)$$

$$(\lambda_1 - \lambda_2)\int_0^\pi u(x)v(x)\,dx = 0$$

Since $\lambda_1 - \lambda_2 \neq 0$ because eigen value are distinct

Hence $\int_0^\pi u(x)v(x)\,dx = 0$ shows u and v are orthogonal

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Lagrange's Identity

Let $u(x)$ and $v(x)$ be two distinct solutions of S-L systems then the following identity must hold

$$uL(v) - vL(u) = \frac{d}{dx}\left\{p(x)(uv' - u'v)\right\}$$

which is called differential form of Lagrang's identity. Also Integral form of identity is

$$\int_a^b [u(x)L(v(x)) - v(x)L(u(x))]\,dx = p(x)[uv' - u'v]_a^b$$

Proof :- Take R.H.S

$$\Rightarrow \frac{d}{dx}[p(x)(uv' - u'v)]$$

$$= p(x)\frac{d}{dx}[uv' - u'v] + [uv' - u'v]\frac{d}{dx}[p(x)]$$

$$= p(x)[uv'' + vu' - u'v' - vu'] + [uv' - u'v]p'(x)$$

$$R.H.S = p(x)[uv'' - vu''] + [uv' - u'v]p'(x) \quad \text{---(1)}$$

Now Take L.H.S

$$= uL(v) - vL(u)$$

$$= u\left[\frac{d}{dx}\left(p(x)\frac{dv}{dx}\right) + q(x)v(x)\right]$$

$$- v\left[\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u(x)\right]$$

$$= u\left[(p(x)v'(x))' + q(x)v(x)\right]$$

$$- v\left[(p(x)u'(x))' + q(x)u(x)\right]$$

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$$= u\left[p(x)v''(x) + v'(x)p'(x) + q(x)v(x) \right]$$

$$-v\left[p(x)u''(x) + u'(x)p'(x) + q(x)u(x) \right]$$

$$= u(x)p(x)v''(x) + u(x)v'(x)p'(x) + u(x)q(x)v(x)$$
$$-v(x)p(x)u''(x) - v(x)u'(x)p'(x) - v(x)q(x)u(x)$$

$$L.H.S = p(x)\left[u(x)v''(x) - v(x)u''(x)\right] + p'(x)\left[u(x)v'(x) - v(x)u'(x)\right]$$

$$L.H.S = p(x)[uv'' - vu''] + p'(x)[uv' - vu'] \quad \text{---} (2)$$

From (1) and (2)

$$uL(v) - vL(u) = \frac{d}{dx}\left[p(x)(uv' - u'v)\right]$$

Hence proved

Reality of Eigen Values

A/2015 , A/2013 , A/2011 , A/2008 , A/2007

Theorem :- The eigen values of an S-L systems (real , periodic) are real.

Proof :- Let u be eigen function corresponding eigen value λ. Then $u(x) \neq 0$, $\forall x \in (a,b)$

$$L(u) + \lambda w(x)u(x) = 0$$

where $L = \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)$

$$L(u) = -\lambda w(x)u(x) \quad \text{---} (1)$$

Suppose on Contrary that λ be complex eigen value.

i-e $\lambda = \lambda_1 + i\lambda_2$

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Taking Conjugate on both sides equation (1)

$$\overline{L(u)} = -\overline{\lambda w(x) v(x)}$$

$$L(\bar{u}) = -\bar{\lambda} w(x) \bar{v}(x) \quad \text{—— (2)}$$

using Lagrange's identity

$$\int_a^b [u L(v) - v L(u)]dx = p(x)[uv' - v'v]_a^b$$

Taking $v = \bar{u}$

$$\int_a^b [u L(\bar{u}) - \bar{u} L(u)]dx = p(x)[u\bar{u}' - v\bar{u}]_a^b$$

$$\int_a^b [u L(\bar{u}) - \bar{u} L(u)]dx = p(b)[u(b)\bar{u}'(b) - v'(b)\bar{u}(b)] - p(a)[u(a)\bar{u}'(a) - v'(a)\bar{u}(a)]$$

using Boundary Conditions

$$\alpha_1 u(a) + \alpha_2 u'(a) = 0 \quad ; \quad \alpha_1 \bar{u}(a) + \alpha_2 \bar{u}'(a) = 0$$

$$\beta_1 u(b) + \beta_2 u'(b) = 0 \quad ; \quad \beta_1 \bar{u}(b) + \beta_2 \bar{u}'(b) = 0$$

$\Rightarrow$

$$u(a) = \frac{-\alpha_2}{\alpha_1} u'(a) \quad ; \quad \bar{u}'(b) = \frac{-\beta_1}{\beta_2} \bar{u}(b)$$

$$u(b) = \frac{-\beta_2}{\beta_1} u'(b) \quad ; \quad \bar{u}'(a) = \frac{-\alpha_1}{\alpha_2} \bar{u}(a)$$

$$\int_a^b [u L(\bar{u}) - \bar{u} L(u)]dx = \left[\frac{-\beta_2}{\beta_1} u'(b) \cdot \frac{-\beta_1}{\beta_2} \bar{u}(b) - u'(b)\bar{u}(b)\right] - p(a)\left[\frac{-\alpha_2}{\alpha_1} u'(a) \cdot \frac{-\alpha_1}{\alpha_2} \bar{u}(a) - u'(a)\bar{u}(a)\right]$$

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$$\int_a^b [uL(\bar{u}) - \bar{u}L(u)]dx = p(b)[u'(b)\bar{u}(b) - u'(b)\bar{u}(b)] - p(a)[u'(a)\bar{u}(a) - u'(a)\bar{u}(a)]$$

$$= p(b)[0] - p(a)[0]$$

$$= 0 - 0$$

$$\int_a^b [uL(\bar{u}) - \bar{u}L(u)]dx = 0 \quad \text{---(3)}$$

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From (1), (2) and (3)

$$\int_a^b [v(-\bar{\lambda}\omega(x)\bar{v}(x)) - \bar{v}(-\lambda\omega(x)v(x))]dx = 0$$

$$\int_a^b \omega(x)\, v(x)\bar{v}(x)\,(-\bar{\lambda}+\lambda)dx = 0$$

$$\int_a^b \omega(x)\,|v(x)|^2\,(-\bar{\lambda}+\lambda)\,dx = 0$$

Since $\omega(x) \neq 0$ and $|v(x)|^2 \neq 0$

So $-\bar{\lambda}+\lambda = 0$

$\Rightarrow \lambda = \bar{\lambda}$

a Contraduction against a fact that λ is a complex number.

Hence eigen values of an S-L system's (regular, periodic) are Real.

Simple Eigen Value :-

An eigen value of regular S-L system is called simple each eigen value ; these coresponed only one eigen solution.

e.g $\lambda_n = C_n \cos \lambda_n x$ is simple Eigen Value

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Theorem :- Eigen values of regular S-L system are simple.

Proof :- On Contrary, suppose $u(x)$ and $v(x)$ be two linearly independent solutions corresponding to same eigen value λ.

$$\therefore L(u) = -\lambda w(x) u(x) \quad \text{---(1)}$$

$$\therefore L(v) = -\lambda w(x) v(x) \quad \text{---(2)}$$

From Lagrange's identity, we have

$$uL(v) - vL(u) = \frac{d}{dx}\{p(x)(uv' - u'v)\}$$

$$uL(v) - vL(u) = \frac{d}{dx}\{p(x) W(u,v)\} \quad \text{---(3)}$$

putting the value of eqn (1) & (2) in (3)

$$u(-\lambda w(x) v(x)) - v(-\lambda w(x) v(x)) = \frac{d}{dx}\{p(x) W(u,v)\}$$

$$\Rightarrow \frac{d}{dx}[p(x) W(u,v)] = 0 \quad \forall x \in [a,b]$$

since $p(x)$ was arbitrary function and $p(x) \neq 0$

For regular S-L system

$$W(u,v) = 0 \quad \forall x \in (a,b)$$

A Contradiction to our supposition that "u" and "v" are independent.

Hence u and v are dependent.

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Normalized Eigen Solution

Example :- (a) Find the eigen value and eigen functions $u'' + \lambda u = 0$

$u(0) = 0$, $u'(1) = 0$

(b) Normalize the eigen function.

Solution :- (a) given that $u'' + \lambda u = 0$

The characteristic equation is

$(D^2 + \lambda) u = 0$

For non-trivial solution $u \neq 0$

So $D^2 + \lambda = 0$

$D = \pm i\sqrt{\lambda}$

General Solution may can be written as

$u(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$ — (1)

$u'(x) = \sqrt{\lambda}[-A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x]$ — (2)

Given B.C's $u(0) = 0$

(1) becomes after applying B.C

$u(0) = A\cos 0 + B\sin 0$

$0 = A(1) + 0$

$\boxed{A = 0}$

Second B.C is $u'(1) = 0$

$u'(1) = \sqrt{\lambda}[-A\sin\sqrt{\lambda}\cdot 1 + B\cos\sqrt{\lambda}\cdot 1]$

Since $A = 0 \Rightarrow 0 = \sqrt{\lambda}[0 + B\cos\sqrt{\lambda}]$

for non-trivial solution $\sqrt{\lambda} \neq 0$

$B\cos\sqrt{\lambda} = 0$

For non-trivial solution $B \neq 0$

$\cos\sqrt{\lambda} = 0$

$\sqrt{\lambda} = \cos^{-1}(0)$

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$\sqrt{\lambda_n} = (2n+1)\frac{\pi}{2}$; $n = 0, 1, 2 ----$

are the eigen values and their corresponding eigen function is $U_n(x) = B_n \sin \sqrt{\lambda_n}\, x$

(b) Normalize

To Normalize $U_n(x)$ we will find the value of B_n such that

$$\langle U_n(x), U_n(x) \rangle = 1$$

i-e

$$\Rightarrow \int_0^1 \omega(x)\, U_n(x) \cdot U_n(x) \cdot dx = 1 \quad \text{---} (3)$$

Given that $u'' + \lambda u = 0$

It can be written as $(1 \cdot u') + 0u(x) + 1 \cdot \lambda u(x) = 0$

Comparing with general form we get $\omega(x) = 1$

(3) becomes $\Rightarrow \int_0^1 1 \cdot [U_n(x)]^2 \cdot dx = 1$

$$\int_0^1 (B_n \sin \sqrt{\lambda} x)^2 \cdot dx = 1$$

$$\int_0^1 B_n^2 \sin^2 \sqrt{\lambda} x = 1$$

$$B_n^2 \int_0^1 \frac{1 - \cos 2\sqrt{\lambda_n}\, x}{2} \cdot dx = 1$$

$$\frac{B_n^2}{2} \int_0^1 (1 - \cos^2 \sqrt{\lambda_n}\, x \cdot dx) = 1$$

$$\frac{B_n^2}{2} \left[x - \frac{\sin 2\sqrt{\lambda} x}{2\sqrt{\lambda}} \right|_0^1 = 1$$

$$\frac{B_n^2}{2} \left[1 - \frac{1}{2\sqrt{\lambda}} \{ \sin 2\sqrt{\lambda} \} \right] = 1$$

$$\frac{B_n^2}{2} - \frac{B_n^2}{4\sqrt{\lambda_n}} \sin 2(2n+1)\frac{\pi}{2} = 1 \quad ; n = 0, 1, 2, 3 \text{---}$$

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Since $\sin(2n+1)\pi = 0 \quad n = 0, 1, 2, 3$

Therefore $\frac{B_n^2}{2} - \frac{B_n^2}{4\sqrt{\lambda}}(0) = 1$

$$\frac{B_n^2}{2} = 1$$

$$B_n^2 = 2 \Rightarrow B_n = \sqrt{2}$$

Hence the normalized eigen Solution is

$$\phi_n(x) = \sqrt{2}(\sin\sqrt{\lambda_n}\, x)$$

Example :- Find the Normalized eigen function for $u'' - 2u' + (1+\lambda)u = 0$

$u(0) = 0 \; ; \; u(1) = 0$

Solution :- Given that

$$u'' - 2u' + (1+\lambda)u = 0$$

The charteristics equation is

$$[D^2 - 2D + (1+\lambda)]u = 0$$

For non-trivial Solution $u \neq 0$

So $D^2 - 2D + (1+\lambda) = 0$

$a = 1 \; , \; b = -2 \; , \; c = 1+\lambda$

$$D = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$D = \frac{2 \pm \sqrt{4 - 4(1)(1+\lambda)}}{2(1)}$$

$$D = \frac{2 \pm \sqrt{4 - 4 - 4\lambda}}{2}$$

$$D = \frac{2 \pm \sqrt{-4\lambda}}{2}$$

$$D = \frac{2 \pm 2i\sqrt{\lambda}}{2} \Rightarrow \frac{2(1 \pm i\sqrt{\lambda})}{2}$$

$$\therefore D = 1 \pm i\sqrt{\lambda}$$

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So the General solution is

$$u(x) = e^{x}[A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x]$$

Apply the boundary conditions

$$u(0)=0$$

$$u(0) = e^{0}[A\cos(0) + B\sin(0)]$$

$$0 = 1\cdot[A(1) + B(0)]$$

$$0 = A + 0 \Rightarrow \boxed{A \neq 0}$$

Now, $u(1)=0$ (Second B.C.)

$$u(1) = e^{1}[A\cos\sqrt{\lambda}\cdot 1 + B\sin\sqrt{\lambda}\cdot 1] = 0$$

Since $A=0$

So $e^{1}[0\cdot\cos\sqrt{\lambda} + B\sin\sqrt{\lambda}] = 0$

$$e[B\sin\sqrt{\lambda}] = 0$$

$$B\sin\sqrt{\lambda} = 0 \quad \because e \neq 0$$

For non-trivial solution $B \neq 0$

$$\sin\sqrt{\lambda} = 0$$

$$\sqrt{\lambda} = \sin^{-1}(0) = n\pi \quad n \in Z$$

$$\lambda = n^2\pi^2$$

are eigen values and their corresponding Eigen function is

$$U_n(x) = e^{x} B_n \sin\sqrt{\lambda_n}\,x$$

Normalized:- $\langle U_n(x), U_n(x)\rangle = 1$

$$\int_a^b w(x)\,U_n(x)\cdot U_n(x) = 1$$

$$\int_a^b w(x)[U_n(x)]^2 = 1 \quad \text{---(i)}$$

For $w(x)$ (weight function) first we convert given D-E into standard S-L equation.

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$$u'' - 2u' + (1+\lambda)u = 0 \quad \text{---(1)}$$

I-F = integration factor $= e^{\int -2\,dx} = e^{-2\int dx} = e^{-2x}$

Multiply I-F with equation (1)

$$e^{-2x}[u'' - 2u' + (1+\lambda)u] = 0$$

$$e^{-2x}u'' - 2e^{-2x}u' + e^{-2x}(1+\lambda)u = 0$$

$$\frac{d}{dx}(e^{-2x}u') + e^{-2x}u(x) + \lambda e^{-2x}u(x) = 0$$

Compare with S-L equation $\frac{d}{dx}(P(x)u') + Q(x)u(x) + \lambda w(x)u(x) = 0$

Here $P(x) = Q(x) = w(x) = e^{-2x}$

put the value of $w(x)$ in (i) the we get

$$\int_0^1 e^{-2x}[B_n^2 e^{2x} \sin^2 \sqrt{\lambda_n}\, x]\,dx = 1$$

$$\int_0^1 (B_n^2 e^{0} \sin^2 \sqrt{\lambda_n}\, x)\,dx = 1$$

$$B_n^2 \int_0^1 (\sin^2 \sqrt{\lambda_n}\, x)\,dx = 1$$

$$B_n^2 \int_0^1 \frac{1 - \cos 2\sqrt{\lambda_n}\, x}{2}\,dx = 1$$

$$\frac{B_n^2}{2}\left[x - \frac{\sin 2\sqrt{\lambda_n}\, x}{2\sqrt{\lambda}}\right]_0^1 = 1$$

$$\frac{B_n^2}{2}\left[1 - \frac{1}{2\sqrt{\lambda}}\{\sin 2\sqrt{\lambda_n}\}\right] = 1$$

$$\frac{B_n^2}{2}\left[1 - \frac{1}{2\sqrt{\lambda}}\{\sin 2(n^2\pi^2)\}\right] = 1 \quad (\sin 2(n^2\pi^2) \to 0)$$

where $n = 0, 1, 2, \ldots$

$$\frac{B_n^2}{2}[1-0] = 1 \Rightarrow B_n = \sqrt{2}$$

Hence the normalized eigen solution is $\phi_n = e^x \sqrt{2} \sin \sqrt{\lambda_n}\, x$

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Power Series Solution of ODE

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For the ODE

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = 0$$

there is a powerfull method for finding the solution if the co-efficients satisfies certain conditions, the a solution is the form of Power Series can be found.

Power Series:-

An infinite series of the form $\sum_{m=0}^{\infty} c_m(x-x_0)^m = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \cdots$ is called a power series with center x_0.

Convergence:-

A numerical values of "x" is subsituted into power series and then power series becomes a series of constant terms Then series is said to converges at x.

Interval of Convergence:-

The set of all valuls of "x" for which the power series converges is said to be interval of convergence.

Radius of Convergence:-

Every interval of convergence has Radius of Convergence.

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For a power series $\sum C_m (x - x_0)^m$ there are three possibilities.

(i) The series convergence only at its center "x_0" Then $R = 0$

(ii) the series converges for all values of "x" satisfy $|x - x_0| < R$ R = Radius

(iii) The series converges for all value of "x" ... Then $R = \infty$

Tests of Convergence :-

(1) Ratio Test :- If $\lim_{n \to \infty} \left| \frac{U_{n+1}}{U_n} \right| |x - x_0| = L$

(i) Series is converges if $L < 1$

(ii) diverges if $L > 1$

(iii) Test fail if $L = 1$

(2) Cauchy's Root Test :-

If $\lim_{n \to \infty} [U_n]^{1/n} = L$

(i) Series is converges if $L < 1$

(ii) Diverges if $L > 1$

(iii) Test fail if $L = 1$

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Question:- Find the Radius and interval of convergence of following series

$$1+\frac{(x+1)}{5}+\frac{(x+1)^2}{5^2}+\frac{(x+1)^3}{5^3}+\cdots$$

Solution:- Given Series

$$\Rightarrow 1+\frac{x+1}{5}+\frac{(x+1)^2}{5^2}+\frac{(x+1)^3}{5^3}+\cdots$$

$$=\sum_{n=0}^{\infty}\frac{(x+1)^n}{6^n}$$

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Center $= x_0 = -1$

Radius of convergence $\rho = \lim_{n\to\infty}\left|\frac{U_n}{U_{n+1}}\right|$

$$\rho = \lim_{n\to\infty}\left|\frac{1/5^n}{1/5^{n+1}}\right|$$

$$\rho = \lim_{n\to\infty}\left|\frac{5^{n+1}}{5^n}\right|$$

$$\rho = \lim_{n\to\infty}|5|$$

$$\boxed{\rho = 5}$$

Interval of Convergence $=]x_0-\rho, x_0+\rho[$

$=]-1-5, -1+5[$

$=]-6, 4[$

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A/2016 {Q1(a)}

Question:- Use the power series method to obtain the general solution for the D-E

$$y'' + y = 0$$

Solution:- Given that $y'' + y = 0$ ——— (A)

let $y = \sum_{n=0}^{\infty} C_n x^n$ ——— (1)

$$y' = \sum_{n=1}^{\infty} C_n n x^{n-1} \quad \text{——— (2)}$$

$$y'' = \sum_{n=2}^{\infty} C_n n(n-1) x^{n-2} \quad \text{——— (3)}$$

Substituting the value of y'' & y in (A)

$$\sum_{n=2}^{\infty} C_n n(n-1) x^{n-2} + \sum_{n=0}^{\infty} C_n x^n = 0 \quad \text{——— (4)}$$

Consider $\sum_{n=2}^{\infty} C_n n(n-1) x^{n-2}$

Let $n - 2 = K$

$n = K + 2$

At $n = 2$, $K = 0$ and at $n = \infty$, $K = \infty$

So $\sum_{n=2}^{\infty} C_n n(n-1) x^{n-2} = \sum_{K=0}^{\infty} C_{K+2} (K+2)(K+1) x^K$

Now Consider $\sum_{n=0}^{\infty} C_n x^n$

And put $n = K$

$$\sum_{n=0}^{\infty} C_n x^n = \sum_{K=0}^{\infty} C_K x^K$$

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Putting these in (4), Then (4) becomes

$$\sum_{K=0}^{\infty} C_{K+2}(K+2)(K+1)x^K + \sum_{K=0}^{\infty} C_K x^K = 0$$

$$\sum_{K=0}^{\infty} \left[C_{K+2}(K+2)(K+1) + C_K\right]x^K = 0$$

Now Comparing the Co-efficient of x^K

$$C_{K+2}(K+2)(K+1) + C_K = 0$$

$$C_{K+2} = \frac{-C_K}{(K+2)(K+1)} \quad \text{——(5)}$$

Put $K = 0$

$$C_2 = \frac{-C_0}{(2)(1)} = \frac{-C_0}{2!}$$

Put $K = 1$

$$C_3 = \frac{-C_1}{(3)(2)} = \frac{-C_1}{3!}$$

Put $K = 2$

$$C_4 = \frac{-C_2}{(4)(3)} = \frac{-1}{(4)(3)}\left[\frac{-C_0}{2!}\right] = \frac{C_0}{4!}$$

Put $K = 3$

$$C_5 = \frac{-C_3}{(5)(4)} = \frac{-1}{(5)(4)}\left[\frac{-C_1}{3!}\right] = \frac{C_1}{5!}$$

Countineuing this process

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AS $y = \sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots$

putting the value of C_2, C_3, C_4, C_5 in above then we get

$$y = C_0 + C_1 x + \left[\frac{-C_0}{2!}\right]x^2 + \left[\frac{-C_1}{3!}\right]x^3 + \left[\frac{C_0}{4!}\right]x^4 + \cdots$$

$$y = C_0\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right] +$$

$$C_1\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right]$$

$$y = C_0 \cos x + C_1 \sin x$$

A/2012

Question:- Use power series method to find linearly independent solution for

$$y'' - xy = 0$$

Solution:- Given that

$$y'' - xy = 0 \quad \text{---(i)}$$

Let $y = \sum_{n=0}^{\infty} C_n x^n$ ---(ii)

$$y' = \sum_{n=1}^{\infty} C_n n x^{n-1} \quad \text{---(iii)}$$

$$y'' = \sum_{n=2}^{\infty} C_n n(n-1) x^{n-2} \quad \text{---(iv)}$$

Substituting these in (i)

$$\sum_{n=2}^{\infty} C_n n(n-1) x^{n-2} = x \sum_{n=0}^{\infty} C_n x^n = 0$$

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$$\sum_{n=2}^{\infty} C_n\, n(n-1)\, x^{n-2} - \sum_{n=0}^{\infty} C_n\, x^{n+1} = 0 \quad \text{—— (v)}$$

We make x^n as the leading power in both series. So it is clear

$$\left\{\begin{aligned} \sum_{n=2}^{\infty} C_n\, n(n-1)\, x^{n-2} &= \sum_{n=0}^{\infty} C_{n+2}\,(n+2)(n+1)\, x^{n} \\ \text{And } \sum_{n=0}^{\infty} C_n\, x^{n+1} &= \sum_{n=1}^{\infty} C_{n-1}\, x^{n} \end{aligned}\right\} \quad \text{—— (vi)}$$

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putting these in (v)

$$\sum_{n=0}^{\infty} C_{n+2}(n+2)(n+1)\,x^{n} - \sum_{n=1}^{\infty} C_{n-1}\,x^{n} = 0$$

$$C_2(2)(1)x^0 + \sum_{n=1}^{\infty} C_{n+2}(n+2)(n+1)x^n - \sum_{n=1}^{\infty} C_{n-1}\,x^n = 0$$

$$2C_2 + \sum_{n=0}^{\infty}\left[(n+2)(n+1)\,C_{n+2} - C_{n-1}\right]x^n = 0$$

Since x^n, $n = 1, 2, 3, 4 - - - - -$ are lineary independent. Therefor we must have

$$C_2 = 0$$

And $(n+2)(n+1)\,C_{n+2} - C_{n-1} = 0$

$$C_{n+2} = \frac{C_{n-1}}{(n+2)(n+1)}$$

For $n = 1$

$$C_3 = \frac{C_0}{3 \cdot 2}$$

For $n = 2$ $\quad C_4 = \dfrac{C_1}{4 \cdot 3}$

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For $n=3 \Rightarrow C_5 = \frac{C_2}{5\cdot4} = \frac{1}{5\cdot4}[0] = 0$

For $n=4 \Rightarrow C_6 = \frac{C_3}{6\cdot5} = \frac{1}{6\cdot5}\left[\frac{C_0}{3\cdot2}\right]$

For $n=5 \Rightarrow C_7 = \frac{C_4}{7\cdot6} = \frac{1}{7\cdot6}\left[\frac{C_1}{4\cdot3}\right] = \frac{C_1}{7\cdot6\cdot4\cdot3}$

For $n=6 \Rightarrow C_8 = \frac{C_5}{8\cdot7} = \frac{1}{8\cdot7}[0] = 0$

As $y = \sum_{n=0}^{\infty} C_n x^n$

$y = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \text{——}$

$y = [C_0 + C_3x^3 + C_6x^6 + C_9x^9] + [C_1x + C_4x^4 + C_7x^7 + \text{——}] + [C_2x^2 + C_5x^5 + C_8x^8 + \text{——}]$

$y = \left[C_0 + \frac{C_0}{3\cdot2}x^3 + \frac{C_0x^6}{6\cdot5\cdot3\cdot2} + \text{——}\right] + \left[C_1x + \frac{C_1x^4}{4\cdot3} + \frac{C_1x^7}{7\cdot6\cdot4\cdot3} + \text{——}\right] + [0+0+0\text{——}]$

$y = C_0\left[1 + \frac{x^3}{3\cdot2} + \frac{x^6}{6\cdot5\cdot3\cdot2} + \text{——}\right] + C_1\left[x + \frac{x^4}{4\cdot3} + \frac{x^7}{7\cdot6\cdot4\cdot3} + \text{——}\right]$

$y = C_0y_1 + C_1y_2$

where $y_1 = 1 + \frac{x^3}{3\cdot2} + \frac{x^6}{6\cdot5\cdot3\cdot2} + \text{——}$

And $y_2 = x + \frac{x^4}{4\cdot3} + \frac{x^7}{7\cdot6\cdot4\cdot3} + \text{——}$

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A/2014

Question:- Find the series solution of $xy'' + y\sin x = 0$ around the Ordinary point $x_0 = 0$

Solution:- The give D-E $xy'' + y\sin x = 0$ in standard form is

$$y'' + y\frac{\sin x}{x} = 0 \quad \text{---------- (1)}$$

Since $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \text{---}$

(1) becomes

$$y'' + y\cdot\frac{1}{x}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \text{---}\right] = 0 \quad \text{(2)}$$

$$y'' + y\left[1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{720} + \text{---}\right] = 0$$

Let $y = \sum_{n=0}^{\infty} c_n x^n$

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}$$

putting these values in (2)

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + \left[1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{720} + \text{---}\right]\sum_{n=0}^{\infty} c_n x^n = 0$$

$$(2)(1)x^0\cdot c_2 + 3\cdot 2x^1 c_3 + 4\cdot 3x^2 c_4 + 5\cdot 4x^3 c_5 + \text{---}$$

$$+\left[1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{120} + \text{---}\right]\left[6x^0 + c_1x^1 + c_2x^2 + c_3x^3 + \text{---}\right] = 0$$

$$2c_2 + 6xc_3 + 12x^2c_4 + 20x^3c_5 + 30x^4c_6 + \text{---}$$

$$+ c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \text{---}$$

$$- \frac{c_0}{6}x^2 - \frac{c_1}{6}x^3 - \frac{c_2}{6}x^4 + \text{---} + \frac{c_0}{120}x^4 + \text{---} = 0$$

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Equating the co-efficients of the terms x^0, x^1, x^2, x^3

$$(2C_1 + C_0)x^0 + (6C_3 + C_1)x + (12C_4 + C_2 - \frac{C_0}{6})x^2$$
$$+ (20C_5 + C_3 - \frac{C_1}{6})x^3 + (30C_6 + C_4 - \frac{C_2}{6} + \frac{C_0}{120})x^4 +$$
$$+ \text{——} + \text{——————————} = 0$$

$$2C_2 + C_0 = 0 \text{ ——————(i)}$$

$$6C_3 + C_1 = 0 \text{ ——————(ii)}$$

$$12C_4 + C_2 - \frac{C_0}{6} = 0 \text{ ——————(iii)}$$

$$20C_5 + C_3 - \frac{C_1}{6} = 0 \text{ ——————(iv)}$$

$$30C_6 + C_4 - \frac{C_2}{6} + \frac{C_0}{120} = 0 \text{ —(v)}$$

From (i) $\boxed{C_2 = -\frac{C_0}{2}}$

From (ii) $\boxed{C_3 = -\frac{C_1}{6}}$

From (iii) $C_4 = \frac{1}{12}\left(\frac{C_0}{6} - C_2\right)$

$$C_4 = \frac{1}{12}\left(\frac{C_0}{6} + \frac{C_2}{2}\right) = \frac{1}{12}\left(\frac{C_0 + 3C_0}{6}\right)$$

$$C_4 = \frac{4C_0}{(12)(6)} = \frac{C_0}{18}$$
$$3$$

$$\boxed{C_4 = \frac{C_0}{18}}$$

From (iv) $20C_5 = \frac{C_1}{6} - C_3$

$$= \frac{C_1}{6} + \frac{C_1}{6} \Rightarrow$$

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$$20C_5 = \frac{2C_1}{6} = \frac{C_1}{3}$$

$$\boxed{C_5 = \frac{C_1}{60}}$$

From (V) $\quad 30C_6 = \frac{C_2}{6} - C_4 - \frac{C_0}{120}$

$$30C_6 = -\frac{C_0}{12} - \frac{C_0}{18} - \frac{C_0}{120}$$

$$30C_6 = \frac{(-30 - 20 - 3)C_0}{360}$$

$$30C_6 = \frac{-53}{360} C_0$$

$$\boxed{C_6 = \frac{-53}{10800} C_0}$$

AS $y = \sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots$

$$y = C_0 + C_1 x + \left(\frac{-C_0}{2}\right)x^2 + \left(\frac{-C_1}{6}\right)x^3 + \left(\frac{C_0}{18}\right)x^4 + \cdots$$

$$\left(\frac{C_1}{60}\right)x^5 + \left(\frac{-53}{10800} C_0\right)x^6 + \cdots$$

$$y = C_0\left[1 - \frac{x^2}{2} + \frac{x^4}{18} - \frac{53}{10800}x^6 + \cdots\right]$$

$$+ C_1\left[x - \frac{x^3}{6} + \frac{x^5}{60} - \cdots\right]$$

$$y = C_0 y_1 + C_1 y_2$$

where $\quad y_1 = 1 - \frac{x^2}{2} + \frac{x^4}{18} - \frac{53}{10800}x^6 + \cdots$

And $\quad y_2 = x - \frac{x^3}{6} + \frac{x^5}{60} - \cdots$

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Question:- Find the power Series solution of $y' - 2xy = 0$ at $x = 0$

Solution:- Given that $y' - 2xy = 0$ — (i)

Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$ — (ii)

$y'(x) = \sum_{n=1}^{\infty} c_n \cdot n x^{n-1}$ — (iii)

Substituting value of y and y' in (i)

$$\sum_{n=1}^{\infty} c_n n x^{n-1} - 2x \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=1}^{\infty} c_n \cdot n x^{n-1} - 2 \sum_{n=0}^{\infty} c_n x^{n+1} = 0 \quad \text{— (iv)}$$

It is clear that

$$\sum_{n=1}^{\infty} c_n n \cdot x^{n-1} = \sum_{n=0}^{\infty} c_{n+1} (n+1) x^n$$

And

$$\sum_{n=0}^{\infty} c_n x^{n+1} = \sum_{n=1}^{\infty} c_{n-1} x^n$$

So (iv) becomes

$$\sum_{n=0}^{\infty} c_{n+1} (n+1) x^n - 2 \sum_{n=1}^{\infty} c_{n-1} x^n = 0$$

$$c_{0+1} (0+1) x^0 + \sum_{n=1}^{\infty} c_{n+1} (n+1) x^n - 2 \sum_{n=1}^{\infty} c_{n-1} x^n = 0$$

$$c_1 x^0 + \sum_{n=1}^{\infty} \left[c_{n+1} (n+1) - 2 c_{n-1} \right] x^n = 0$$

Comparing the coefficient of x^0

$$\boxed{c_1 = 0}$$

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Comparing the Co-efficient of x^n ; $n \geq 1$

$$(n+1)C_{n+1} - 2C_{n-1} = 0$$

$$C_{n+1} = \frac{2C_{n-1}}{(n+1)}$$

put $n = 1$

$$C_2 = \frac{2C_0}{2} = C_0$$

put $n = 2$

$$C_3 = \frac{2}{3} C_1 = 0 \qquad \boxed{\because C_1 = 0}$$

put $n = 3$

$$C_4 = \frac{2}{4} C_2 = \frac{1}{2} C_2 = \frac{1}{2} C_0$$

put $n = 4$

$$C_5 = \frac{2}{5} C_3 = 0$$

$$C_{2n} = \frac{1}{n!} C_0$$

$$C_{2n+1} = 0$$

$$\text{As } y = \sum_{n=0}^{\infty} C_n x^n$$

$$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \text{---}$$

$$\text{---} + C_{2n-1} x^{2n-1} + C_{2n} x^{2n} + \text{---}$$

Putting the values $C_1, C_2, C_3 \text{---} C_{2n}$ in above

$$y = C_0 + C_0 x^2 + \frac{1}{2!} C_0 x^4 + \frac{1}{3!} C_0 x^6 + \text{---} \frac{1}{n!} C_0 x^{2n} + \text{---}$$

$$= C_0 \left[1 + x^2 + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \text{---} \right]$$

$$y = C_0 e^{x^2}$$

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Question:- $y'' - 2xy' + 3y = 0$ at $x_0 = 1$

Solution:- Given that $y'' - 2xy' + 3y = 0$ —(1)

This is second order linear homogeneous ordinary DE with variable co-effcients.

Here every value of "x" is an ordinary First we corresponds the independent variable such that $t = 0$" is the center of expansion

Let $t = x - 1$

Then $t_0 = 0$, is the Center of expansion

Now we transform (1) such that "t" is the independent variable.

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{d}{dt}\right)$$

$$= \frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} = \frac{d^2y}{dt^2}$$

On making substitution in (1)

$$y_{tt} - 2(t+1)y_t + 3y = 0 \quad —(2)$$

let $y = \sum_{n=0}^{\infty} c_n t^n$ —(3)

Diff w.r.t "t"

$$y_t = \sum_{n=1}^{\infty} c_n\, n t^{n-1} \quad —(4)$$

$$y_{tt} = \sum_{n=2}^{\infty} c_n\, n(n-1)\, t^{n-2} \quad —(5)$$

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substituting ③, ④ & ⑤ in ②

$$\sum_{n=2}^{\infty} C_n\, n(n-1)\, t^{n-2} - 2(t+1)\sum_{n=1}^{\infty} C_n\, n t^{n-1} + 3\sum_{n=0}^{\infty} C_n t^n = 0$$

$$\sum_{n=2}^{\infty} C_n\, n(n-1)\, t^{n-2} - 2\sum_{n=1}^{\infty} C_n \cdot n t^n - 2\sum_{n=1}^{\infty} C_n \cdot n t^{n-1} + 3\sum_{n=0}^{\infty} C_n t^n \quad \text{⑥}$$

It is clear ⑦

$$\sum_{n=2}^{\infty} C_n \cdot n(n-1)\, t^{n-2} = \sum_{n=0}^{\infty} C_{n+2}\,(n+1)(n+2)\, t^n$$

$$\sum_{n=1}^{\infty} C_n\, n \cdot t^{n-1} = \sum_{n=0}^{\infty} C_{n+1}\,(n+1)\, t^n$$

putting ⑦ in ⑥

$$\sum_{n=0}^{\infty} C_{n+2}\,(n+1)(n+2)\, t^n - 2\sum_{n=1}^{\infty} C_n \cdot n t^n - 2\sum_{n=0}^{\infty} C_{n+1}(n+1) t^n + 3\sum_{n=0}^{\infty} C_n t^n = 0$$

$$C_2\,(1)(2)\cdot t^0 + \sum_{n=1}^{\infty} C_{n+2}(n+1)(n+2)\, t^n - 2\sum_{n=1}^{\infty} C_n \cdot n t^n$$

$$-2\, C_1\,(1)\cdot t^0 - 2\sum_{n=0}^{\infty} C_{n+1}(n+1) t^n + 3C_0 t^0 + 3\sum_{n=1}^{\infty} C_n t^n = 0$$

$$2C_2 - 2C_1 + 3C_0 + \sum_{n=1}^{\infty}\left[(n+1)(n+2)C_{n+2} - 2nC_n - 2(n+1)C_{n+1} + 3C_n\right] t^n = 0$$

$$2C_2 - 2C_1 + 3C_0 + \sum_{n=1}^{\infty}\left[(n+1)(n+2)C_{n+2} - 2(n+1)C_{n+1} + (3-2n)C_n\right] t^n = 0$$

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Since x_n are linearly independent $n=0,1,2,3$ —

So we must have

$$2C_2 - 2C_1 + 3C_0 = 0$$

$$2C_2 = 2C_1 - 3C_0$$

$$C_2 = C_1 - \frac{3}{2}C_0$$

The recurrence relation is

$$(n+1)(n+2)C_{n+2} - 2(n+1)C_{n+1} + (3-2n)C_n = 0$$

$n = 1, 2, 3$ ——

$$(n+1)(n+2)C_{n+2} = 2(n+1)C_{n+1} + (2n-3)C_n$$

$$C_{n+2} = \frac{2(n+1)C_{n+1} + (2n-3)C_n}{(n+1)(n+2)}$$

$$C_{n+2} = \frac{2(n+1)C_{n+1}}{(n+1)(n+2)} + \frac{(2n-3)C_n}{(n+1)(n+2)}$$

$$C_{n+2} = \frac{2C_{n+1}}{(n+2)} + \frac{(2n-3)C_n}{(n+1)(n+2)}$$

For $n = 1$

$$C_3 = \frac{2C_2}{3} - \frac{C_1}{2\cdot 3}$$

$$C_3 = \frac{2}{3}\left[C_1 - \frac{3}{2}C_0\right] - \frac{C_1}{6}$$

$$C_3 = \frac{2}{3}C_1 - C_0 - \frac{C_1}{6}$$

$$C_3 = \left(\frac{2}{3} - \frac{1}{6}\right)C_1 - C_0$$

$$C_3 = \frac{C_1}{2} - C_0$$

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For $n=2$

$$C_4 = \frac{2C_3}{3\cdot 4} + \frac{C_2}{3\cdot 4}$$

$$= \frac{C_3}{3\cdot 2} + \frac{C_2}{12}$$

$$= \frac{1}{6}\left[\frac{C_1}{2} - C_0\right] + \frac{1}{12}\left[C_1 - \frac{3}{2}C_0\right]$$

$$= \frac{C_1}{12} - \frac{C_0}{6} + \frac{C_1}{12} - \frac{C_0}{8}$$

$$= \left(\frac{1}{12}+\frac{1}{12}\right)C_1 - \left(\frac{1}{6}+\frac{1}{8}\right)C_0$$

$$= \frac{1}{6}C_1 - \frac{7}{24}C_0$$

Since $y = \sum_{n=0}^{\infty} C_n t^n = C_0 + C_1 t + C_2 t^2 + C_3 t^3 + \cdots$

$$y = C_0 + C_1 t + \left(C_1 - \frac{3}{2}C_0\right)t^2 + \left(\frac{C_1}{2} - C_0\right)t^3 + \left[\frac{C_1}{6} - \frac{7}{24}\right]t^4 + \cdots$$

$$y = C_0\left[1 - \frac{3}{2}t^2 - t^3 + \cdots\right] + C_1\left[t + t^2 + \frac{t^3}{2} + \cdots\right]$$

As $t = x-1$

So $y = C_0\left[1 - \frac{3}{2}(x-1)^2 - (x-1)^3 + \cdots\right] + C_1\left[(x-1) + (x-1)^2 + \frac{(x-1)^3}{2} + \cdots\right]$

$y = C_0 y_1 + C_1 y_2$

where $y_1 = 1 - \frac{3}{2}(x-1)^2 - (x-1)^3 + \cdots$

$y_2 = (x-1) + (x-1)^2 + \frac{(x-1)^3}{2} + \cdots$

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A/2015, A/2013

Question:- Define ordinary and singular points of DE $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ when the a singular point is said to be regular and irregular?

Find regular and irregular singular points of the D-E $(x^2-4)^2y'' + (x-2)y' + y = 0$

Answer:-

Ordinary Points:-

Consider a linear Second order D-E

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \quad \text{---(1)}$$

A point x_0 is called ordinary point if $a_2(x_0) \neq 0$

Alternatively:- Take equation (i)

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

Divided by $a_2(x)$

$$y'' + \frac{a_1(x)}{a_2(x)}y' + \frac{a_0(x)}{a_2(x)}y = 0$$

$$y'' + P(x)y' + Q(x)y = 0 \quad \text{---(ii)}$$

where $P(x) = \frac{a_1(x)}{a_2(x)}$ and $Q(x) = \frac{a_0(x)}{a_2(x)}$

Now A point x_0 is an Ordinary point of equation (ii) if $P(x)$ and $Q(x)$ are analytic at x_0.

Analytic function:- A function f is analytic at x_0 if it has a Taylor series expansion that converges to f in some interval about x_0.

$$f(x) = \sum_{n}^{\infty} f_n (x-x_0)^n$$

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Singular Points :-

Consider eqn (ii) from previous page

$$y'' + P(x)y' + Q(x)y = 0$$

where $P(x) = \frac{a_1(x)}{a_2(x)}$ and $Q(x) = \frac{a_0(x)}{a_2(x)}$

All those points where $P(x)$ and $Q(x)$ are not analytic are called singular points.

Regular Singular Points :-

A singular point $x = x_0$ of D.E $y'' + P(x)y' + Q(x)y = 0$ is said to be regular singular point if both $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ are analytic at $x = x_0$

Irregular Singular Points :-

A singular point $x = x_0$ of D-E $y'' + P(x)y' + Q(x)y = 0$ is said to be irregular singular point if both $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ are not analytic at $x = x_0$

Example :- $(x^2 - 4)^2 y'' + (x-2)y' + y = 0$

$$y'' + \frac{(x-2)}{(x^2-4)^2} y' + \frac{1}{(x^2-4)^2} y = 0$$

$$P(x) = \frac{x-2}{(x^2-4)^2} = \frac{(x-2)}{(x+2)^2(x-2)^2} = \frac{1}{(x+2)^2(x-2)}$$

$$Q(x) = \frac{1}{(x^2-4)^2}$$

Hence $x = 2$ is Regular and $x = -2$ is irregular point.

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FROBENIUS METHOD

If x_0 is a regular singular point of the ordinary Differential Equation

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

Solution may be found by the Frobenius method or by expansion in a Laurent Series.

In Frobenius method, assume a Solution of the form

$$y = x^k \sum_{n=0}^{\infty} c_n x^n$$

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$$y = \sum_{n=0}^{\infty} c_n x^{n+k}$$

$$y' = \sum_{n=0}^{\infty} c_n (n+k) x^{n+k-1}$$

$$y'' = \sum_{n=0}^{\infty} c_n (n+k)(n+k-1) x^{n+k-2}$$

Now plug y back into the ODE and group the co-efficients by power to obtain a recursion formula for the c_nth term, and then write the Series expansion in terms of the c_n's. Equating the c_0 term to zero will produce the, So-called "Indicial Equation", which give the allowed value of "k" in Series Expension.

INDICIAL EQUATION

An indicial equation, also called a characteristic equation, is a recurrence equation obtained during

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application of the Frobenius method of Solving a Second-order ordinary differential equation:

The indical equation is obtained by nothing that, by Definition, the lowest order term x^k (that corresponding to $n=0$) must have a co-efficient of zero.

(1):- If the two roots are equal. Then only one Solution can be obtained

(2):- If the two roots differ by a non-integer Then two Solutions can be obtained.

(3) If the two roots differ by an integer. then larger will yield a solution the smaller may or may not.

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A/2013

Question:- Using the Frobenius method, Find two linearly independent solution about the singular point $x_0=0$ of D-E

$$2xy''+(1+x)y'+y=0$$

Solution:- Given $2xy''+(1+x)y'+y=0$ —(i)

Let $y=\sum_{n=0}^{\infty} C_n x^{n+r}$

$$y'=\sum_{n=0}^{\infty} C_n (n+r) x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty} C_n (n+r)(n+r-1) x^{n+r-2}$$

Substituting these in (i)

$$2x\left[\sum_{n=0}^{\infty} C_n (n+r)(n+r-1) x^{n+r-2}\right]+(1+x)\left[\sum_{n=0}^{\infty} C_n (n+r) x^{n+r}\right]$$

$$+\left[\sum_{n=0}^{\infty} C_n x^{n+r}\right]=0$$

$$2\sum_{n=0}^{\infty} C_n (n+r)(n+r-1) x^{n+r-1}+\sum_{n=0}^{\infty} C_n (n+r) x^{n+r-1}$$

$$+\sum_{n=0}^{\infty} C_n (n+r) x^{n+r}+\sum_{n=0}^{\infty} C_n x^{n+r}=0$$

$$2C_0(r)(r-1)x^{r-1}+2\sum_{n=1}^{\infty} C_n (n+r)(n+r-1) x^{n+r-1}$$

$$+C_0(r)(x^{r-1})+\sum_{n=1}^{\infty} C_n (n+r) x^{n+r-1}+\sum_{n=0}^{\infty} C_n (n+r+1) x^{n+r}=0$$

$$2C_0(r)(r-1)x^{r-1}+C_0(r)(x^{r-1})+\sum_{n=1}^{\infty} C_n (n+r)\Big[$$

$$2n+2r-2+1\Big] x^{n+r-1}+\sum_{n=0}^{\infty} C_n (n+r+1) x^{n+r}=0$$

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$$r C_0 (2r-2+1) x^{r-1} + \sum_{n=0}^{\infty} C_{n+1} (n+1+r)[2n+2+2r -2+1] x^{n+r} + \sum_{n=0}^{\infty} C_n (n+1+r) x^{n+r} = 0$$

Comparing the Co-efficient of x^{r-1}

$$C_0\, r(2r-1) = 0$$

Since $C_0 \neq 0$ $\quad r(2r-1)=0$

$r = 0$ or $2r-1=0 \Rightarrow r = 1/2$

Now comparing the co-efficent of x^{n+r}

$$C_{n+1}(n+1+r)(2n+2r+1) + C_n(n+1+r) = 0$$

$$(n+1+r)[C_{n+1}(2n+2r+1) + C_n] = 0$$

$$C_{n+1}(2n+2r+1) + C_n = 0$$

$$C_{n+1} = \frac{-C_n}{2n+2r+1} \qquad (2)$$

For $r = 0$

$$C_{n+1} = \frac{-C_n}{2n+1}$$

Put $n=0$

$$C_1 = -\frac{C_0}{1} = -C_0 \Rightarrow \boxed{C_1 = -C_0}$$

Put $n=1$

$$C_2 = \frac{-C_1}{3} = -\frac{1}{3}(-C_0) = \frac{C_0}{3}$$

$$\boxed{C_2 = C_0/3}$$

Put $n=2$ $\quad C_3 = \frac{-C_2}{5} = -\frac{1}{5}\left[\frac{C_0}{3}\right] = \frac{-C_0}{15} \Rightarrow \boxed{C_3 = -\frac{C_0}{15}}$

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Put $n=3$ $\quad C_4 = \frac{-C_3}{7} = -\frac{1}{7}\left[\frac{-C_0}{15}\right] = \frac{C_0}{105}$

$$\boxed{C_4 = C_0/105}$$

Hence $\quad y_1 = \sum_{n=0}^{\infty} C_n x^{n+r}$

For $r=0$

$$y_1 = \sum_{n=0}^{\infty} C_n x^n \quad \text{---------- i}$$

$$y_1 = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \text{-----}$$

$$y_1 = C_0 - C_0 x + \frac{C_0}{3} x^2 - \frac{C_0}{15} x^3 + \frac{C_0}{105} x^4 \text{ ----------}$$

$$\boxed{y_1 = C_0\left[1 - x + \frac{x^2}{3} - \frac{x^3}{15} + \frac{x^4}{105} \text{ ----------}\right]}$$

Now $\underline{r = 1/2}$ in ②

$$C_{n+1} = \frac{-C_n}{2n+2(1/2)+1} = \frac{-C_n}{2n+2}$$

$$C_{n+1} = \frac{-C_n}{2(n+1)}$$

Put $n=0$

$$\boxed{C_1 = -\frac{C_0}{2}}$$

Put $n=1$

$$C_2 = -\frac{C_1}{4} = -\frac{1}{4}\left[\frac{-C_0}{2}\right] = \frac{C_0}{8}$$

$$\boxed{C_2 = C_0/8}$$

Put $n=2$

$$C_3 = \frac{-C_2}{6} = -\frac{1}{6}\left[\frac{C_0}{8}\right] = \frac{-C_0}{48}$$

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$$\boxed{C_3 = -C_0/48}$$

As $y_2 = \sum_{n=0}^{\infty} C_n x^{n+r}$ For $r = 1/2$

$$y_2 = \sum_{n=0}^{\infty} C_n x^{n+1/2}$$

$$y_2 = C_0 x^{1/2} + C_1 x^{3/2} + C_2 x^{5/2} + C_3 x^{7/2} + C_4 x^{9/2} + \cdots$$

$$y_2 = C_0 x^{1/2} - \frac{C_0}{2} x^{3/2} + \frac{C_0}{8} x^{5/2} - \frac{C_0}{48} x^{7/2} + \cdots$$

$$\boxed{y_2 = C_0 \left[x^{1/2} - \frac{x^{3/2}}{2} + \frac{x^{5/2}}{8} - \frac{x^{7/2}}{48} + \cdots \right]}$$

A/2014

Question:- Using frobenius method to obtain two linearly independent series solution a regular singular point $x_0 = 0$ of the D-E

$$2xy'' - y' + 2y = 0$$

Solution:- Given that

$$2xy'' - y' + 2y = 0 \quad \text{---(i)}$$

Let $y = \sum_{n=0}^{\infty} C_n x^{n+r}$

$$y' = \sum_{n=0}^{\infty} C_n (n+r) x^{n+r-1}$$

and $$y'' = \sum_{n=0}^{\infty} C_n (n+r)(n+r-1) x^{n+r-2}$$

Substituing these in (i)

$$2x \left[\sum_{n=0}^{\infty} C_n (n+r)(n+r-1) x^{n+r-2} \right] - \left[\sum_{n=0}^{\infty} C_n (n+r) x^{n+r-1} \right] + 2 \left[\sum_{n=0}^{\infty} C_n x^{n+r} \right] = 0$$

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$$2C_n(n+\gamma)(n+\gamma-1)x^{n+\gamma-1} - \sum_{n=0}^{\infty} C_n(n+\gamma)x^{n+\gamma-1} + 2\sum_{n=0}^{\infty} C_n x^{n+\gamma} = 0$$

$$\sum_{n=0}^{\infty} C_n(n+\gamma)\left[2(n+\gamma-1)-1\right]x^{n+\gamma-1} + 2\sum_{n=0}^{\infty} C_n x^{n+\gamma} = 0$$

$$\sum_{n=0}^{\infty} C_n(n+\gamma)\left[2n+2\gamma-3\right]x^{n+\gamma-1} + 2\sum_{n=0}^{\infty} C_n x^{n+\gamma} = 0$$

$$C_0\,\gamma(2\gamma-3)x^{\gamma-1} + \sum_{n=1}^{\infty} C_n(n+\gamma)(2n+2\gamma-3)x^{n+\gamma-1} + 2\sum_{n=0}^{\infty} C_n x^{n+\gamma} = 0$$

$$C_0\,\gamma(2\gamma-3)x^{\gamma-1} + \sum_{n=0}^{\infty} C_{n+1}(n+\gamma+1)(2n+2\gamma-1)x^{n+\gamma} + 2\sum_{n=0}^{\infty} C_n x^{n+\gamma} = 0$$

$$C_0\,\gamma(2\gamma-3)x^{\gamma-1} + \sum_{n=0}^{\infty}\left[C_{n+1}(n+\gamma+1)(2n+2\gamma-1)+2C_n\right]x^{n+\gamma} = 0$$

Comparing the co-effiecnt of $x^{\gamma-1}$

$$C_0(\gamma)(2\gamma-3) = 0$$

$C_0 \neq 0$ $\boxed{\gamma = 0}$; $2\gamma - 3 = 0 \Rightarrow \boxed{\gamma = 3/2}$

Now Comparing the Co-effcient of $x^{n+\gamma}$

$$C_{n+1}(n+\gamma+1)(2n+2\gamma-1) + 2C_n = 0$$

$$C_{n+1} = \frac{-2C_n}{(n+\gamma+1)(2n+2\gamma-1)} \quad \text{---(2)}$$

For $\gamma = 0$ (2) becomes

$$C_{n+1} = \frac{-2C_n}{(n+1)(2n-1)}$$

Put $n = 0$

$$C_1 = \frac{-2C_0}{(1)(-1)} = 2C_0$$

Put $n = 1$

$$C_2 = \frac{-2C_1}{(2)(1)} = -C_1 = -2C_0$$

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Put $n=2$

$$C_3 = \frac{-2C_2}{(3)(3)} = \frac{-2C_2}{9} = -\frac{2}{9}[-2C] = \frac{4}{9}C_0$$

Put $n=3$

$$C_4 = \frac{-2C_3}{(4)(5)} = \frac{-C_3}{10} = -\frac{1}{10}\left[\frac{4}{9}C_0\right] = -\frac{2}{45}C_0$$

As $y = \sum_{n=0}^{\infty} C_n x^{n+r}$

For $r=0$

$$y = \sum_{n=0}^{\infty} C_n x^{n} = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots$$

$$y = C_0 + 2C_0 x - 2C_0 x^2 + \frac{4}{9}C_0 x^3 - \frac{2}{45}C_0 x^4 + \cdots$$

$$\boxed{y = C_0\left[1 + 2x - 2x^2 + \frac{4}{9}x^3 - \frac{2}{45}x^4 + \cdots\right]}$$

For $r = 3/2$ $\Rightarrow$ (2) becomes

$$C_{n+1} = \frac{-2C_n}{\left(n+\frac{3}{2}+1\right)\left(2n+2\left(\frac{3}{2}\right)-1\right)}$$

$$C_{n+1} = \frac{-2C_n}{(n+5/2)(2n+2)}$$

$$C_{n+1} = \frac{-2C_n}{(n+5/2)(2)(n+1)}$$

$$C_{n+1} = \frac{-C_n}{(n+5/2)(n+1)}$$

Put $n=0$

$$C_1 = \frac{-C_0}{(5/2)(1)} = -\frac{2}{5}C_0$$

Put $n=1$

$$C_2 = \frac{-C_1}{\left(1+\frac{5}{2}\right)(2)} = \frac{-C_1}{\left(\frac{7}{2}\right)(2)} = -\frac{C_1}{7}$$

$$C_2 = -\frac{1}{7}\left[-\frac{2}{5}C_0\right] = \frac{2}{35}C_0$$

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Put $n=2$

$$C_3 = \frac{-C_2}{(2+5/2)(2+1)} = \frac{-C_2}{(\frac{9}{2})(3)} = -\frac{2}{27}C_2$$

$$C_3 = -\frac{2}{27}\left[\frac{2}{35}C_0\right] = -\frac{4}{945}C_0$$

As $y = \sum_{n=0}^{\infty} C_n x^{n+r}$ For $r = 3/2$

$$y = \sum_{n=0}^{\infty} C_n x^{n+3/2}$$

$$y = x^{3/2}\left[C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \text{——}\right]$$

$$y = x^{3/2}\left[C_0 - \frac{2}{5}C_0 x + \frac{2}{35}C_0 x^2 - \frac{4}{945}C_0 x^3 + \text{——}\right]$$

$$\boxed{y = C_0 x^{3/2}\left[1 - \frac{2}{5}x + \frac{2}{35}x^2 - \frac{4}{945}x^3 + \text{——}\right]}$$

Question: Solve $3xy'' + y' - y = 0$ around $x=0$ using Frobenius method.

Solution:- Given $3xy'' + y' - y = 0$ —— (1)

That's clear $x=0$ is a regular singular pt.

Let $y = \sum_{n=0}^{\infty} C_n x^{n+r}$

$$y' = \sum_{n=0}^{\infty} C_n (n+r) x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} C_n (n+r)(n+r-1) x^{n+r-2}$$

Substituting in (1)

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$$3x\left[\sum_{n=0}^{\infty} C_n (n+\gamma)(n+\gamma-1) x^{n+\gamma-2}\right] + \sum_{n=0}^{\infty} C_n (n+\gamma) x^{n+\gamma-1}$$

$$- \sum_{n=0}^{\infty} C_n x^{n+\gamma} = 0$$

$$3x\, C_0 (\gamma)(\gamma-1) x^{\gamma-2} + \sum_{n=1}^{\infty} 3x\, C_n (n+\gamma)(n+\gamma-1) x^{n+\gamma-2}$$

$$+ C_0 (\gamma) x^{\gamma-1} + \sum_{n=1}^{\infty} C_n (n+\gamma) x^{n+\gamma-1} - \sum_{n=0}^{\infty} C_n x^{n+\gamma} = 0$$

$$3 C_0 (\gamma)(\gamma-1) x^{\gamma-1} + C_0 (\gamma) x^{\gamma-1} + \sum_{n=1}^{\infty} \Big[3C_n (n+\gamma)(n+\gamma-1)$$

$$+ C_n (n+\gamma) \Big] x^{n+\gamma-1} - \sum_{n=0}^{\infty} C_n x^{n+\gamma} = 0$$

Comparing the co-efficient of $x^{\gamma-1}$

$$[3\gamma(\gamma-1) + \gamma] C_0 = 0$$

$$[3\gamma^2 - 3\gamma + \gamma] C_0 = 0$$

$C_0 \neq 0 \qquad 3\gamma^2 - 2\gamma = 0$

$$\gamma(3\gamma - 2) = 0$$

$\gamma = 0 \quad ; \quad 3\gamma - 2 = 0$

$\gamma = 2/3$

And

$$\sum_{n=1}^{\infty} \Big[3C_n (n+\gamma)(n+\gamma-1) + C_n (n+\gamma) \Big] x^{n+\gamma-1} - \sum_{n=0}^{\infty} C_n x^{n+\gamma} = 0$$

$$\sum_{n=0}^{\infty} \Big[3C_{n+1} (n+1+\gamma)(n+\gamma) + C_{n+1} (n+1+\gamma) \Big] x^{n+\gamma} - \sum_{n=0}^{\infty} C_n x^{n+\gamma} = 0$$

$$\sum_{n=0}^{\infty} \Big[3C_{n+1} (n+1+\gamma)(n+\gamma) + C_{n+1} (n+1+\gamma) - C_n \Big] x^{n+\gamma} = 0$$

Now Comparing the Co-efficients of $x^{n+\gamma}$

$$3C_{n+1} (n+1+\gamma)(n+\gamma) + C_{n+1} (n+1+\gamma) - C_n = 0$$

$$C_{n+1} (n+1+\gamma) [3n + 3\gamma + 1] - C_n = 0$$

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$$C_{n+1} = \frac{C_n}{(n+1+r)(3n+3r+1)}$$

For $r=0$ $\quad C_{n+1} = \dfrac{C_n}{(n+1)(3n+1)}$

put $n=0 \Rightarrow C_1 = \dfrac{C_0}{(1)(1)} = C_0$

put $n=1 \Rightarrow C_2 = \dfrac{C_1}{(2)(4)} = \dfrac{C_0}{2!\cdot 4}$

put $n=2 \Rightarrow C_3 = \dfrac{C_2}{(3)(7)} = \dfrac{C_0}{3\cdot 7\cdot 2!4} = \dfrac{C_0}{3!\cdot 1\cdot 4\cdot 7}$

put $n=3 \Rightarrow C_4 = \dfrac{C_3}{(4)(10)} = \dfrac{C_0}{4\cdot 10\cdot 3!\cdot 1\cdot 4\cdot 7} = \dfrac{C_0}{4!\cdot 1\cdot 4\cdot 7\cdot 10}$

$$C_n = \frac{C_0}{n!\ 1\cdot 4\cdot 7 \cdots (3n-2)}$$

For $r=\frac{2}{3}$ $\quad C_{n+1} = \dfrac{C_n}{\left(n+\frac{2}{3}+1\right)(3n+2+1)}$

$$C_{n+1} = \frac{C_n}{\left[\frac{3n+2+3}{3}\right][3n+3]}$$

$$C_{n+1} = \frac{C_n}{(3n+5)(n+1)}$$

put $n=0$

$$C_1 = \frac{C_0}{(5)(1)} = \frac{C_0}{5}$$

put $n=1$ $\quad C_2 = \dfrac{C_1}{(8)(2)} = \dfrac{C_0}{2!\cdot 5\cdot 8}$

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For $n=2$

$$C_2 = \frac{C_2}{(11)(3)} = \frac{C_0}{3\cdot 2!\, 5\cdot 8\cdot 11} = \frac{C_0}{3!\, 5\cdot 8\cdot 11}$$

Similarly

$$C_n = \frac{C_0}{n!\ 5\cdot 8\cdot 11 \cdots (3n+2)}$$

As $y_1 = \sum_{n=0}^{\infty} C_n x^{n+\gamma}$ ∴

For $\gamma=0 \Rightarrow y_1 = \sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots$

$$\boxed{y_1 = \sum_{n=0}^{\infty} \frac{C_0}{n!\ 1\cdot 4\cdot 7 \cdots (3n-2)} x^n}$$

And $y_2 = \sum_{n=0}^{\infty} C_n x^{n+\gamma}$

For $\gamma = 2/3$

$$y_2 = \sum_{n=0}^{\infty} C_n x^n \cdot x^{2/3}$$

$$\boxed{y_2 = \sum_{n=0}^{\infty} \frac{C_0}{n!\ 5\cdot 8\cdot 11 \cdots (3n+2)} x^n \cdot x^{2/3}}$$

Both series are independent $(\because \gamma_1 - \gamma_2 = 2/3)$

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Question:- Solve $2xy'' + y' + xy = 0$ using frobenius method about $x_0 = 0$

Solution:- Given $2xy'' + y' + xy = 0$ — (1)

Let $y = \sum_{n=0}^{\infty} C_n x^{n+\gamma}$

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$$y' = \sum_{n=0}^{\infty} C_n (n+\gamma)\, x^{n+\gamma-1}$$

$$y'' = \sum_{n=0}^{\infty} C_n (n+\gamma)(n+\gamma-1)\, x^{n+\gamma-2}$$

Substituting these in (1)

$$2x\left[\sum_{n=0}^{\infty} C_n (n+\gamma)(n+\gamma-1)\, x^{n+\gamma-2}\right] + \left[\sum_{n=0}^{\infty} C_0 (n+\gamma) x^{n+\gamma-1}\right]$$

$$+ x\left[\sum_{n=0}^{\infty} C_n x^{n+\gamma}\right] = 0$$

$$2\sum_{n=0}^{\infty} C_n (n+\gamma)(n+\gamma-1) x^{n+\gamma-1} + \sum_{n=0}^{\infty} C_n (n+\gamma) x^{n+\gamma-1} + \sum_{n=0}^{\infty} C_n x^{n+\gamma+1} = 0$$

$$\sum_{n=0}^{\infty} \left[2C_n (n+\gamma)(n+\gamma-1) + C_n (n+\gamma)\right] x^{n+\gamma-1} + \sum_{n=0}^{\infty} C_n x^{n+\gamma+1} = 0$$

$$\sum_{n=0}^{\infty} \left[2n+2\gamma-2+1\right] C_n (n+\gamma) x^{n+\gamma-1} + \sum_{n=0}^{\infty} C_n x^{n+\gamma+1} = 0$$

$$\sum_{n=0}^{\infty} \left[2n+2\gamma-1\right](n+\gamma) C_n x^{n+\gamma-1} + \sum_{n=0}^{\infty} C_n x^{n+\gamma+1} = 0$$

$$(2\gamma-1)(\gamma) C_0 x^{\gamma-1} + \sum_{n=1}^{\infty} \left[2n+2\gamma-1\right](n+\gamma) C_n x^{n+\gamma-1}$$

$$+ \sum_{n=1}^{\infty} C_{n-1} x^{n+\gamma} = 0$$

$$(2\gamma-1)(\gamma) C_0 x^{\gamma-1} + \sum_{n=0}^{\infty} \left[2(n+1)+2\gamma-1\right](n+1+\gamma) C_n x^{n+\gamma}$$

$$+ \sum_{n=1}^{\infty} C_{n-1} x^{n+\gamma} = 0$$

$$(2\gamma-1)(\gamma) C_0 x^{\gamma-1} + \sum_{n=1}^{\infty} \left[(2n+2\gamma+1)(n+1+\gamma) C_{n+1} + C_{n-1}\right] x^{n+\gamma} = 0$$

Comparing the Co-efficient of $x^{\gamma-1}$

$$(2\gamma-1)(\gamma) C_0 = 0$$

since $C_0 \neq 0$ so $\boxed{\gamma = 0}$ & $2\gamma - 1 = 0 \Rightarrow \boxed{\gamma = 1/2}$

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Now Comparing the Co-efficient of $x^{n+\gamma}$

$$(2n+2\gamma+1)(n+1+\gamma)C_{n+1} + C_{n-1} = 0$$

$$C_{n+1} = \frac{-C_{n-1}}{(2n+2\gamma+1)(n+1+\gamma)} \quad (2)$$

For $\gamma = 0$ (2) becomes

$$C_{n+1} = \frac{-C_{n-1}}{(2n+1)(n+1)}$$

put $n=1$

$$C_2 = \frac{-C_0}{3\cdot 2}$$

put $n=2 \Rightarrow C_3 = \frac{-C_1}{5\cdot 3} = 0$ (in Since $C_1 = 0$)

put $n=3 \Rightarrow C_4 = \frac{-C_2}{7\cdot 4} = \frac{-1}{7\cdot 4}\left[\frac{-C_0}{3\cdot 2}\right] = \frac{C_0}{7\cdot 4\cdot 3\cdot 2}$

AS $y_1 = \sum_{n=0}^{\infty} C_n x^{n+\gamma}$

For $\gamma = 0$

$$y_1 = \sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + \cdots$$

$$y_1 = C_0 + 0 - \frac{C_0}{2\cdot 3}x^2 + 0 + \frac{C_0}{2\cdot 4\cdot 3\cdot 7}x^4 + \cdots$$

$$y_1 = C_0\left[1 - \frac{x^2}{3!} + \frac{x^4}{4!\cdot 7} + \cdots\right]$$

For $\gamma = 1/2$ (2) becomes

$$C_{n+1} = \frac{-C_{n-1}}{\left[2n+2\left(\frac{1}{2}\right)+1\right]\left[n+1+\frac{1}{2}\right]}$$

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$$C_{n+1} = \frac{-C_{n-1}}{(2n+2)\frac{(2n+3)}{2}} = \frac{-C_{n-1}}{(n+1)(2n+3)}$$

Put $n=1 \Rightarrow C_2 = \frac{-C_0}{2\cdot 5}$

Put $n=2 \Rightarrow C_3 = \frac{-C_1}{3\cdot 7} = 0 \quad (\because C_1 = 0)$

Put $n=3 \Rightarrow C_4 = \frac{-C_2}{(4)(9)} = \frac{-1}{(4)(9)}\left(\frac{-C_0}{2\cdot 5}\right) = \frac{C_0}{2\cdot 4\cdot 5\cdot 9}$

As $y = \sum_{n=0}^{\infty} C_n x^{n+r}$

For $r = 1/2$

$$y_2 = \sum_{n=0}^{\infty} C_n x^{n+1/2}$$

$$y_2 = \sum_{n=0}^{\infty} C_n x^n \cdot x^{1/2}$$

$$y_2 = x^{1/2} \sum_{n=0}^{\infty} C_n x^n$$

$$y_2 = x^{1/2}\left[C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots\right]$$

$$y_2 = x^{1/2}\left[C_0 + 0 - \frac{C_0}{(2)(5)} x^2 + 0 + \frac{C_0}{(2)(4)(5)(9)} x^4 + \cdots\right]$$

$$y_2 = x^{1/2} C_0\left[1 - \frac{x^2}{10} + \frac{x^4}{360} - \cdots\right]$$

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A/2015

Question :- Use the method of Frobenius to obtain Two linearly independent series solutions about the singular point $x_0 = 0$ of the D-E

$$x y'' + (1-x) y' - y = 0$$

Solution :- Given D-E

$$x y'' + (1-x) y' - y = 0 \quad \text{---(1)}$$

Let $y = \sum_{n=0}^{\infty} C_n x^{n+r}$

$$y' = \sum_{n=0}^{\infty} C_n (n+r) x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} C_n (n+r)(n+r-1) x^{n+r-2}$$

Substituting these in (1)

$$x\left[\sum_{n=0}^{\infty} C_n (n+r)(n+r-1) x^{n+r-2}\right] + (1-x)\left[\sum_{n=0}^{\infty} C_n (n+r) x^{n+r-1}\right] - \sum_{n=0}^{\infty} C_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} C_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=0}^{\infty} C_n (n+r) x^{n+r-1} - \sum_{n=0}^{\infty} C_n (n+r) x^{n+r} - \sum_{n=0}^{\infty} C_n x^{n+r} = 0$$

$$C_0 (r)(r-1) x^{r-1} + \sum_{n=1}^{\infty} C_n (n+r)(n+r-1) x^{n+r-1} + C_0 (r) x^{r-1} + \sum_{n=1}^{\infty} C_n (n+r)(x^{n+r-1}) - \sum_{n=0}^{\infty} C_n (n+r+1) x^{n+r} = 0$$

$$C_0 r(r-1+1) x^{r-1} + \sum_{n=0}^{\infty} C_{n+1} (n+r+1)(n+r) x^{n+1} + \sum_{n=0}^{\infty} C_{n+1} (n+1+r) x^{n+r} - \sum_{n=0}^{\infty} C_n (n+r+1) x^{n+r} = 0$$

Comparing the Co-efficient of x^{r-1}

$$C_0 r^2 = 0$$

For non-trivial Solution $C_0 \neq 0$

$$r^2 = 0$$

$$r = 0$$

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Now Compering the co-efficient of "x^{n+r}"

$$C_{n+1}(n+1+r)(n+r) + C_{n+1}(n+1+r) - C_n(n+1+r) = 0$$

$$(n+1+r)[C_{n+1}(n+r) + C_{n+1} - C_n] = 0$$

Since $n+1+r \neq 0$

$$C_{n+1}(n+r+1) - C_n = 0$$

$$C_{n+1} = \frac{C_n}{n+r+1} \quad \text{— (2)}$$

For $r=0$ (2) becomes

$$C_{n+1} = \frac{C_n}{n+1}$$

put $n=0$ $\quad C_1 = C_0$

put $n=1 \Rightarrow C_2 = \frac{C_1}{2} = \frac{C_0}{2} = \frac{C_0}{2!}$

put $n=2 \Rightarrow C_3 = \frac{C_2}{3} = \frac{1}{3} \frac{C_0}{2!} = \frac{C_0}{3!}$

put $n=3 \Rightarrow C_4 = \frac{C_3}{4} = \frac{C_0}{4 \cdot 3!} = \frac{C_0}{4!}$

Similarly Continuing process

$$C_n = \frac{C_0}{n!}$$

AS $\quad y_1 = \sum_{n=0}^{\infty} C_n x^{n+r}$

For $r=0$

$$y = \sum_{n=0}^{\infty} C_n x^n$$

$$y_1 = \sum_{n=0}^{\infty} \frac{C_0}{n!} x^n$$

$$y_1 = C_0 \left[\sum_{n=0}^{\infty} \frac{x^n}{n!}\right]$$

$$y_1 = C_0 e^x \quad \text{since } C_0 = 1$$

$$y_1 = e^x$$

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Now to find the second linearly independent solution of (1) we use

$$y_2 = y_1 \ln x + w(x)$$

where $w(x) = \sum_{n=0}^{\infty} a_n x^n$

Then (1) can be written as

$$x y_2'' + (1-x) y_2' - y_2 = 0$$

$$x\left[y_1'' \ln x + \frac{2y_1'}{x} + \frac{y_1}{x} - \frac{y_1}{x^2} + w''\right] + (1-x)\left[y_1' \ln x + \frac{y_1}{x} + w'\right] - [y_1 \ln x + w(x)] = 0$$

$$x\left[y_1'' \ln x + \frac{2}{x} y_1' - \frac{y_1}{x^2} + w''\right] + (1-x)\left[y_1' \ln x + \frac{y_1}{x} + w'\right] - (y_1 \ln x + w) = 0$$

$$x y_1'' \ln x + 2y_1' - \frac{y_1}{x} + x w'' + y_1' \ln x + \frac{y_1}{x} + w' - x y_1' \ln x - y_1 - x w' - y_1 \ln x - w = 0$$

$$\left[x y_1'' + (1-x) y_1' - y_1\right] \ln x + 2y_1' - y_1 + x w'' + w' - x w' - w = 0$$

Since y_1 is Solution of (1). So terms with $\ln x$ vanishes and we get

$$x w'' + (1-x) w' - w = y_1 - 2y_1'$$

$$x\left[\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}\right] + (1-x)\left[\sum_{n=0}^{\infty} n a_n x^{n-1}\right] - \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} C_n x^n - 2\left[\sum_{n=0}^{\infty} n C_n x^{n-1}\right]$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-1} + \sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} C_n x^n - 2\sum_{n=0}^{\infty} n C_n x^{n-1}$$

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$$\sum_{n=0}^{\infty}[n^2-n+n]a_n x^{n-1} - \sum_{n=0}^{\infty}(n+1)a_n x^n$$

$$= \sum_{n=0}^{\infty} C_n x^n - 2\sum_{n=0}^{\infty} n C_n x^{n-1}$$

$$\sum_{n=0}^{\infty} n^2 a_n x^{n-1} - \sum_{n=0}^{\infty}(n+1)a_n x^n = \sum_{n=0}^{\infty} C_n x^n - 2\sum_{n=0}^{\infty} n C_n x^{n-1}$$

$$\sum_{n=-1}^{\infty}(n+1)^2 a_{n+1} x^n - \sum_{n=0}^{\infty}(n+1)a_n x^n = \sum_{n=0}^{\infty} C_n x^n - 2\sum_{n=-1}^{\infty}(n+1)C_{(n+1)} x^n$$

$$\sum_{n=0}^{\infty}\left[(n+1)^2 a_{n+1} - (n+1)a_n\right]x^n = \sum_{n=0}^{\infty}\left[C_n - 2(n+1)C_{n+1}\right]x^n$$

$$\sum_{n=0}^{\infty}\left[(n+1)^2 a_{n+1} - (n+1)a_n - C_n + 2(n+1)C_{n+1}\right]x^n = 0$$

AS x^n are linearly independent $n=0, 1, 2, 3$ ——

$$(n+1)^2 a_{n+1} - (n+1)a_n - C_n + 2(n+1)C_{n+1} = 0$$

$$(n+1)^2 a_{n+1} = (n+1)a_n + C_n - 2(n+1)C_{n+1} = 0$$

$$a_{n+1} = \frac{(n+1)a_n + C_n - 2(n+1)C_{n+1}}{(n+1)^2}$$

$$a_{n+1} = \frac{a_n}{(n+1)} + \frac{C_n}{(n+1)^2} - \frac{2C_{n+1}}{(n+1)}$$

for $n=0$

$$a_1 = \frac{a_0}{1} + \frac{C_0}{1^2} - \frac{2C_1}{1}$$

$a_1 = a_0 + C_0 - 2C_0$ $\quad \because C_1 = C_0 = 1$

$a_1 = a_0 - C_0$

$a_1 = a_0 - 1$

$a_1 = 1 - 1$ $\quad$ Take $a_0 = 1$

$a_1 = 0$

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For $n=1 \Rightarrow a_2 = \frac{a_1}{2} + \frac{C_1}{(1+1)^2} - \frac{2C_2}{(1+1)}$

$a_2 = \frac{C_1}{2^2} - C_2 \quad \because a_1 = 0$

$a_2 = \frac{1}{4} - \frac{1}{2}$

$a_2 = -1/4$

For $n=2$

$a_3 = \frac{a_2}{3} + \frac{C_2}{3^2} - \frac{2C_3}{3}$

$a_3 = \frac{-1}{4}\left(\frac{1}{3}\right) + \frac{1}{9}\left(\frac{C_0}{1}\right) - \frac{2}{3}\left(\frac{C_0}{6}\right)$

$a_3 = \frac{-1}{12} + \frac{1}{18} - \frac{1}{9} \quad \because C_0 = 1$

$a_3 = -5/36$

As $w = \sum_{n=0}^{\infty} a_n x^n$

$w = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$w = 1 + 0 - \frac{x^2}{4} - \frac{5}{36}x^3 + \cdots$

$w = 1 - \frac{x^2}{4} - \frac{5}{36}x^3 + \cdots$

As $y_2 = y_1 \ln x + w$

So $y_2 = [e^x \cdot \ln x] + \left[1 - \frac{x^2}{4} - \frac{5x^3}{36} + \cdots\right]$

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BESSEL, LEGENDRE And HYPERGEOMETRIC FUNCTIONS

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Bessel's Equation:-

An equation of the form $x^2y'' + xy' + (x^2 - \nu^2)y = 0$ where $\nu \geq 0$ is called Bessel's D-E, and particular solutions of this equation are called Bessel's functions of order ν.

A/2016

Series Solution of Bessel's D-E

$x^2y'' + xy' + (x^2 - \nu^2)y = 0$ — Bessel's D-E

Let $y = \sum_{n=0}^{\infty} C_n x^{n+r}$ ...

$y' = \sum_{n=0}^{\infty} C_n (n+r) x^{n+r-1}$

$y'' = \sum_{n=0}^{\infty} C_n (n+r)(n+r-1) x^{n+r-2}$

Substituting these in Bessel's D-E

$$x^2\left[\sum_{n=0}^{\infty} C_n (n+r)(n+r-1) x^{n+r-2}\right] + x\left[\sum_{n=0}^{\infty} C_n (n+r) x^{n+r-1}\right] + (x^2 - \nu^2)\sum_{n=0}^{\infty} C_n x^{n+r} = 0 \quad (1)$$

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$$\sum_{n=0}^{\infty} C_n (n+\gamma)(n+\gamma-1) x^{n+\gamma} + \sum_{n=0}^{\infty} C_n (n+\gamma) x^{n+\gamma} + \sum_{n=0}^{\infty} C_n x^{n+\gamma+2} - \gamma^2 \sum_{n=0}^{\infty} C_n x^{n+\gamma} = 0 \quad \text{---(2)}$$

$$C_0(\gamma)(\gamma-1)x^{\gamma} + \sum_{n=1}^{\infty} C_n (n+\gamma)(n+\gamma-1) x^{n+\gamma} + C_0(\gamma) x^{\gamma} + \sum_{n=1}^{\infty} C_n (n+\gamma) x^{n+\gamma} - \gamma^2 C_0 x^{\gamma} - \gamma^2 \sum_{n=1}^{\infty} C_n x^{n+\gamma} + \sum_{n=0}^{\infty} C_n x^{n+\gamma+2} = 0$$

$$\left[C_0 \gamma(\gamma-1) + \gamma C_0 - \gamma^2 C_0\right] x^{\gamma} + \sum_{n=1}^{\infty} \left[C_n (n+\gamma)(n+\gamma-1) + C_n (n+\gamma) - \gamma^2 C_n\right] x^{n+\gamma} + \sum_{n=0}^{\infty} C_n x^{n+\gamma+2} = 0$$

Comparing the co-efficient of x^{γ}

$$C_0 \gamma(\gamma-1) + \gamma C_0 - \gamma^2 C_0 = 0$$

$$(\gamma^2 - \gamma + \gamma - \gamma^2) C_0 = 0$$

$$(\gamma^2 - \gamma^2) C_0 = 0$$

Since $C_0 \neq 0$ $\quad \gamma^2 - \gamma^2 = 0$

$$\gamma^2 = \gamma^2$$

$$\gamma = \pm \gamma$$

When $\gamma = \gamma$ in (2) we get

$$\sum_{n=0}^{\infty} C_n (n+\gamma)(n+\gamma-1) x^{n+\gamma} + \sum_{n=0}^{\infty} C_n (n+\gamma) x^{n+\gamma} + \sum_{n=0}^{\infty} C_n x^{n+\gamma+2} - \gamma^2 \sum_{n=0}^{\infty} C_n x^{n+\gamma} = 0$$

$$\sum_{n=0}^{\infty} C_n \left[(n+\gamma)(n+\gamma-1) + (n+\gamma) - \gamma^2\right] x^{n+\gamma} + \sum_{n=2}^{\infty} C_{n-2} x^{n+\gamma} = 0$$

$$\sum_{n=0}^{\infty} C_n \left[(n+\gamma)(n+\gamma-1+1) - \gamma^2\right] x^{n+\gamma} + \sum_{n=2}^{\infty} C_{n-2} x^{n+\gamma} = 0$$

$$\sum_{n=0}^{\infty} C_n \left[(n+\gamma)^2 - \gamma^2\right] x^{n+\gamma} + \sum_{n=2}^{\infty} C_{n-2} x^{n+\gamma} = 0$$

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$$C_0[\nu^2-\nu^2]x^{\nu}+C_1[(1+\nu)^2-\nu^2]x^{1+\nu}+\sum_{n=2}^{\infty}C_n[(n+\nu)^2-\nu^2]x^{n+\nu}$$

$$+\sum_{n=2}^{\infty}C_{n-2}x^{n+\nu}=0$$

$$C_1[1+\nu^2+2\nu-\nu^2]x^{1+\nu}+\sum_{n=2}^{\infty}[C_n\{(n+\nu)^2-\nu^2\}+C_{n-2}]x^{n+\nu}=0$$

$$C_1[1+2\nu]x^{1+\nu}+\sum_{n=2}^{\infty}[C_n\{(n+\nu)^2-\nu^2\}+C_{n-2}]x^{n+\nu}=0$$

comparing the Co-efficient of $x^{1+\nu}$

$$C_1(1+2\nu)=0$$

Since $\nu>0$ There for $1+2\nu\neq 0$.

So $\boxed{C_1=0}$

Now comparing the Co-efficient of $x^{n+\nu}$

$$C_n[(n+\nu)^2-\nu^2]+C_{n-2}=0$$

$$C_n(n^2+\nu^2+2n\nu-\nu^2)+C_{n-2}=0$$

$$C_n(n^2+2n\nu)=-C_{n-2}$$

$$C_n=\frac{-C_{n-2}}{n(n+2\nu)}$$

Put $n=2 \Rightarrow C_2=\frac{-C_0}{2(2+2\nu)}=\frac{-C_0}{2^2(1+\nu)}$

Put $n=4 \Rightarrow C_4=\frac{-C_2}{4(4+2\nu)}=\frac{C_0}{2\cdot 2^2(2+\nu)\cdot 2^2(1+\nu)}$

$$C_4=\frac{C_0}{2^4\cdot 2!(1+\nu)(2+\nu)}$$

Put $n=6$

$$C_6=\frac{-C_4}{6(6+2\nu)}$$

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$$C_6 = \frac{-C_4}{6\cdot 2(3+\nu)} = \frac{-C_4}{12\ (3+\nu)}$$

$$C_6 = \frac{-C_0}{4\cdot 3(3+\nu)\cdot 2^4\cdot 2!\,(1+\nu)(2+\nu)}$$

$$C_6 = \frac{-C_0}{2^6\cdot 3!\,(1+\nu)(2+\nu)(3+\nu)}$$

Continuing this process we get

$$C_{2n} = \frac{(-1)^n C_0}{2^{2n}\cdot n!\,(1+\nu)(2+\nu)(3+\nu)\cdots(n+\nu)}$$

If we chose $C_0 = \frac{1}{2^\nu \Gamma(1+\nu)}$

Then

$$C_2 = \frac{-1}{1\cdot 2^2(1+\nu)\cdot 2^\nu \Gamma(1+\nu)}$$

$$C_2 = \frac{-1}{2^{2+\nu}\,1!\,[1+\nu\,\Gamma(1+\nu)]}$$

$$C_2 = \frac{-1}{2^{2+\nu}\cdot 1!\,\Gamma(1+\nu+1)}$$

$$C_4 = \frac{1}{4\cdot 2\cdot 2!\,(1+\nu)(2+\nu)\cdot 2^\nu \Gamma(1+\nu)}$$

$$C_4 = \frac{1}{2^{4+\nu}\cdot 2!\,(2+\nu)(1+\nu)\Gamma(1+\nu)}$$

$$C_4 = \frac{1}{2^{4+\nu}\,2!\,(2+\nu)\,\Gamma(2+\nu)}$$

$$C_4 = \frac{1}{2^{4+\nu}\,2!\,\Gamma(2+\nu+1)} = \frac{1}{2^{4+\nu}\cdot 2!\,\Gamma(1+\nu+2)}$$

Property of Gamma function
(i) $\Gamma(n+1) = n!$
(ii) $n\,\Gamma(n) = \Gamma(n+1)$

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$$C_6 = \frac{-1}{2^{6+\nu} \cdot 3!\, \Gamma(1+\nu+3)}$$

Simi Lorly $C_{2n} = \dfrac{(-1)^n}{2^{(2n+\nu)} \cdot n!\, \Gamma(1+\nu+n)}$

$$J_\nu(x) = y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+\nu} \cdot n!\, \Gamma(1+\nu+n)} (x)^{2n+\nu}$$

For $\nu = -\nu$

$$J_{-\nu}(x) = y_2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n-\nu}\, n!\, \Gamma(1-\nu+n)} x^{2n-\nu}$$

It can be noted that the series solutions are linearly independent (one cannot be expressed in terms of second).

Case I = If ν is not integer or zero, Then the complete solution of (1) is

$$y = A J_\nu(x) + B J_{-\nu}(x)$$

Case II = If $\nu = 0$ then $y_1 = y_2$ and complete solution of Bessel's Function of order zero.

Case III = If ν is positive integer then y_2 is not solution of Bessel's D-E. And y_1 fails to give a solution for negative values of ν. Let us find out the general solution when ν is an integer.

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Bessel's Function of Second Kind

If ν is not integer then the function defined by

$$Y_\nu(x) = \frac{\cos(\nu\pi)\, J_\nu(x) - J_{-\nu}(x)}{\sin(\nu x)}$$

and the function $J_\nu(x)$ are linearly independent solutions of Bessel's equation for the value of "ν". And the general solution is then

$$y = A J_\nu(x) + B Y_\nu(x)$$

Parametric Bessel Equation

As Bessel equation is $x^2y'' + xy' + (x^2 - \nu^2)y = 0$

Replace x by λx we get

$$x^2 y'' + x y' + (\lambda^2 x^2 - \nu^2)y = 0$$

The general solution is

$$Y = A J_\nu(\lambda x) + B Y_\nu(\lambda x)$$

Example:- $x^2y'' + xy' + (4x^2 - 1)y = 0$ — (1)

Let $2x = \rho \Rightarrow x = \frac{\rho}{2}$

$$2 \cdot dx = d\rho$$

$$2 = \frac{d\rho}{dx}$$

$$\frac{dy}{dx} = \frac{dy}{d\rho}\cdot\frac{d\rho}{dx} = 2 \cdot \frac{dy}{d\rho}$$

And $$\frac{d^2y}{dx^2} = 2 \cdot \frac{d^2y}{d\rho^2}\cdot\frac{d\rho}{dx}$$

$$= 2 \cdot \frac{d^2y}{d\rho^2}(2)$$

$$= 4\, d^2y/d\rho^2$$

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putting these in (1)

$$\left[\frac{\rho}{2}\right]^2 \cdot 4 \cdot \frac{d^2y}{d\rho^2} + \left[\frac{\rho}{2}\right] 2 \frac{dy}{d\rho} + (\rho^2 - 1) y = 0$$

$$\frac{\rho^2}{4} \cdot 4 \cdot \frac{d^2y}{d\rho^2} + \frac{\rho}{2} \cdot 2 \cdot \frac{dy}{d\rho} + (\rho^2 - 1) y = 1$$

$$\rho^2 \cdot \frac{d^2y}{d\rho^2} + \rho \cdot \frac{dy}{d\rho} + (\rho^2 - 1) y = 1$$

$$\rho^2 y'' + \rho y' + (\rho^2 - 1) y = 1$$

So $\nu^2 = 1 \Rightarrow \nu = +1$

$r_1 - r_2 = 1 - (-1) = 2$,

$$y(\rho) = A J_1(\rho) + B Y_1(\rho)$$

$$y(x) = A J_1(2x) + B Y_1(2x)$$

Example:- If m is +ve integer then show

$$J_{-m}(x) = (-1)^m J_m(x)$$

Solution:- As we know that

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+\nu} \cdot n! \, \Gamma(1+\nu+n)} \cdot x^{2n+\nu}$$

put $\nu = m$

$$J_{-m}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \, \Gamma(1-m+n)} \left(\frac{x}{2}\right)^{2n-m}$$

As $\Gamma(n-m+1)$ is defined for $n - m + 1 > 0$

$n > m - 1$

i.e $n \geq m$

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So $$J_{-m}(x) = \sum_{n=m}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n-m+1)} \left[\frac{x}{2}\right]^{2n-m}$$

Let $n - m = \gamma$

when $n = m \longrightarrow \gamma = 0$

when $n \to \infty$ then $\gamma \to \infty$

$$= \sum_{\gamma=0}^{\infty} \frac{(-1)^{m+\gamma}}{(m+\gamma)!\,\Gamma(\gamma+1)} \left[\frac{x}{2}\right]^{2m+2\gamma-m}$$

Note that, $\Gamma(\gamma+1) = \gamma!$

And $(m+\gamma)! = \Gamma(m+\gamma+1)$

$$J_{-m}(x) = \sum_{\gamma=0}^{\infty} \frac{(-1)^m \cdot (-1)^{\gamma}}{\Gamma(m+\gamma+1) \cdot \gamma!} \left[\frac{x}{2}\right]^{2\gamma+m}$$

$$J_{-m}(x) = (-1)^m \sum_{\gamma=0}^{\infty} \frac{(-1)^{\gamma}}{\gamma!\,\Gamma(m+\gamma+1)} \left(\frac{x}{2}\right)^{2\gamma+m}$$

$J_{-m}(x) = (-1)^m J_m(x)$ Hence prove

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A/2007

Question :- Prove that the Generating function of Bessel's function $J_n(x)$ of integral order is given by $e^{\frac{x}{2}\left[t - \frac{1}{t}\right]} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$

where $t \in z$

Proof :- Consider $e^{\frac{x}{2}\left[t - \frac{1}{t}\right]} = e^{\frac{xt}{2}} \cdot e^{-\frac{x}{2t}}$

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As we know that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

So $e^{\frac{xt}{2} - \frac{x}{2t}} = e^{\frac{xt}{2}} \cdot e^{-\frac{x}{2t}} = \sum_{r=0}^{\infty} \frac{\left(\frac{xt}{2}\right)^r}{r!} \cdot \sum_{s=0}^{\infty} \frac{\left(-\frac{x}{2t}\right)^s}{s!}$

$= \sum_{r=0}^{\infty} \frac{\left(\frac{x}{2}\right)^r t^r}{r!} \cdot \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^s t^{-s}}{s!}$

$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{r+s} t^{r-s}}{r!\, s!}$

put $r - s = n$

or $r = n + s$

$= \sum_{n=-\infty}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{n+s+s} t^n}{s!\,(n+s)!}$

Note :- $(n+s)! = \Gamma(n+s+1)$

$= \sum_{n=-\infty}^{\infty} \left[\sum_{s=0}^{\infty} \frac{(-1)^s \left[\frac{x}{2}\right]^{2s+n}}{s!\,\Gamma(n+s+1)} \right] t^n$

$e^{\frac{x}{2}\left[t - \frac{1}{t}\right]} = \sum_{n=-\infty}^{\infty} J_n(x) \cdot t^n$ Hence proove

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A/2011

RECURRENCE RELATION FOR BESSEL'S FUNCTION

(1) $x J_\nu'(x) = \nu J_\nu(x) - x J_{\nu+1}(x)$

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Proof:- Take L.H.S $= x J_\nu'(x)$

$$= x \frac{d}{dx}\left[J_\nu(x)\right]$$

$$= x \frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+\nu} \cdot n! \, \Gamma(1+\nu+n)} x^{2n+\nu}\right]$$

$$= x\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+\nu} \cdot n! \, \Gamma(1+\nu+n)} (2n+\nu) x^{2n+\nu-1}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\nu) x^{2n+\nu-1+1}}{2^{2n+\nu} \cdot n! \, \Gamma(1+\nu+n)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\nu)}{2^{2n+\nu} \cdot n! \, \Gamma(1+\nu+n)} \cdot x^{2n+\nu}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\nu)}{n! \, \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2n)}{n! \, \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} + \sum_{n=0}^{\infty} \frac{(-1)^n \cdot \nu \, (x/2)^{2n+\nu}}{n! \, \Gamma(1+\nu+n)}$$

$$= 2\sum_{n=0}^{\infty} \frac{(-1)^n \cdot n}{n(n-1)! \, \Gamma(1+\nu+n)} \left[\frac{x}{2}\right]^{2n+\nu} + \nu J_\nu(x)$$

put $n-1 = r$

$n = r+1$

$$= 2\sum_{r=0}^{\infty} \frac{(-1)^{r+1}}{(n-1)! \, \Gamma(1+\nu+r+1)} \left(\frac{x}{2}\right)^{2r+2+\nu} + \nu J_\nu(x)$$

$$= 2\sum_{r=0}^{\infty} \frac{(-1)(-1)^r}{r! \, \Gamma(2+\nu+r)} \left(\frac{x}{2}\right)^{2r+\nu+1+1} + \nu J_\nu(x)$$

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$$= -x\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(1+\nu+r+1)}\left(\frac{x}{2}\right)^{2r+\nu+1}\cdot\left(\frac{x}{2}\right) + \nu J_\nu(x)$$

$$= -x\sum_{r=0}^{\infty} \frac{(-1)^r}{2^{2r+\nu+1}\, r!\,\Gamma(1+\nu+r+1)}(x)^{2r+\nu+1} + \nu J_\nu(x)$$

$$= -x J_{\nu+1}(x) + \nu J_\nu(x)$$

$$= \nu J_\nu(x) - x J_{\nu+1}(x) = R.H.S$$

Hence proved

(2) $x J_\nu'(x) = x J_{\nu-1}(x) - \nu J_\nu(x)$

Take $L.H.S = x J_\nu'(x)$

$$= x\frac{d}{dx}[J_\nu(x)]$$

$$= x\frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+\nu}\cdot n!\,\Gamma(1+\nu+n)}(x^{2n+\nu})\right]$$

$$= x\sum_{n=0}^{\infty} \frac{(-1)^n (2n+\nu)\, x^{2n+\nu-1}}{2^{2n+\nu}\cdot n!\,\Gamma(1+\nu+n)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\nu)\, x^{2n+\nu}}{2^{2n+\nu}\cdot n!\,\Gamma(1+\nu+n)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+2\nu-\nu)}{n!\,\Gamma(1+\nu+n)}\cdot\left(\frac{x}{2}\right)^{2n+\nu}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+2\nu)}{n!\,\Gamma(1+\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu} - \sum_{n=0}^{\infty} \frac{(-1)^n (\nu)}{n!\,\Gamma(1+\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu}$$

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$$= 2\sum_{n=0}^{\infty} \frac{(-1)^n (n+\nu)}{n!\,(n+\nu)\,\Gamma(n+\nu)} \left(\frac{x}{2}\right)^{2n+\nu} - \nu J_\nu(x)$$

$$= x\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+(\nu-1)+1)} \left(\frac{x}{2}\right)^{2n+\nu-1} \left(\frac{x}{2}\right) - \nu J_\nu(x)$$

$$= x\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma[n+(\nu-1)+1]} \left(\frac{x}{2}\right)^{2n+\nu-1} - \nu J_\nu(x)$$

$$= x J_{\nu-1}(x) - \nu J_\nu(x) = \text{R.H.S}$$

Hence prove

(3) $2J_\nu'(x) = J_{\nu-1}(x) - J_{\nu+1}(x)$

As we know from recurrence relation ① & ②

$$xJ_\nu'(x) = \nu J_\nu(x) - xJ_{\nu+1}(x) \quad \text{—— ①}$$

$$xJ_\nu'(x) = xJ_{\nu-1}(x) - \nu J_\nu(x) \quad \text{—— ②}$$

Adding ① & ②

$$xJ_\nu'(x) = \nu J_\nu(x) - xJ_{\nu+1}(x)$$

$$xJ_\nu'(x) = -\nu J_\nu(x) + xJ_{\nu-1}(x)$$

$$2xJ_\nu'(x) = xJ_{\nu-1}(x) - xJ_{\nu+1}(x)$$

$$2xJ_\nu'(x) = x\left[J_{\nu-1}(x) - J_{\nu+1}(x)\right]$$

$$2J_\nu'(x) = J_{\nu-1}(x) - J_{\nu+1}(x)$$

Hence prove

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8/2014

(4):- $J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) - J_{\nu-1}(x) \quad ; \nu > 1$

Proof :- As we know from recurrence relation (1) & (2)

$$x J_\nu'(x) = \nu J_\nu(x) - x J_{\nu+1}(x) \quad \text{---(1)}$$

$$x J_\nu'(x) = x J_{\nu-1}(x) - \nu J_\nu(x) \quad \text{---(2)}$$

Subtracting (1) & (2), we get

$$x J_\nu'(x) = \nu J_\nu(x) - x J_{\nu+1}(x)$$

$$-x J_\nu'(x) = -x J_{\nu-1}(x) + \nu J_\nu(x)$$

$$0 = 2\nu J_\nu(x) - x J_{\nu-1}(x) - x J_{\nu+1}(x)$$

$$x J_{\nu+1}(x) = 2\nu J_\nu(x) - x J_{\nu-1}(x)$$

$$J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) - J_{\nu-1}(x)$$ Hence prove

* * * *

(5) $- \frac{d}{dx}\{x^\nu J_\nu(x)\} = x^\nu J_{\nu-1}(x)$

$$L.H.S = \frac{d}{dx}\{x^\nu J_\nu(x)\}$$

$$= \nu x^{\nu-1} J_\nu(x) + x^\nu J_\nu'(x) \quad \text{---(A)}$$

From relation (4) $J_\nu(x) = \frac{x}{2\nu}\left[J_{\nu-1}(x) + J_{\nu+1}(x)\right]$

From relation (3) $J_\nu'(x) = \frac{1}{2}\left[J_{\nu-1}(x) - J_{\nu+1}(x)\right]$

using these in (A) then, we get

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$$= x^{\nu-1}\cdot\frac{x}{2\nu}\left[J_{\nu-1}(x)+J_{\nu+1}(x)\right]+\frac{x^{\nu}}{2}\left[J_{\nu-1}(x)-J_{\nu+1}(x)\right]$$

$$=\frac{x^{\nu}}{2}\left[J_{\nu-1}(x)+J_{\nu+1}(x)\right]+\frac{x^{\nu}}{2}\left[J_{\nu-1}(x)-J_{\nu+1}(x)\right]$$

$$=\frac{x^{\nu}}{2}\left[J_{\nu-1}(x)+J_{\nu+1}(x)+J_{\nu-1}(x)-J_{\nu+1}(x)\right]$$

$$=\frac{x^{\nu}}{2}\left[2J_{\nu-1}(x)\right]$$

$$=x^{\nu}J_{\nu-1}(x) \quad =R.H.S$$

(6) $\frac{d}{dx}\left\{x^{-\nu}J_{\nu}(x)\right\}=-x^{-\nu}J_{\nu+1}(x)$

$$L.H.S=\frac{d}{dx}\left\{x^{-\nu}J_{\nu}(x)\right\}$$

$$=-\nu x^{-\nu-1}J_{\nu}(x)+x^{-\nu}J_{\nu}'(x) \quad \text{---(A)}$$

From relation (4) & (3)

$$J_{\nu}(x)=\frac{x}{2\nu}\left[J_{\nu-1}(x)+J_{\nu+1}(x)\right]$$

$$J_{\nu}'(x)=\frac{1}{2}\left[J_{\nu-1}(x)-J_{\nu+1}(x)\right]$$

putting these in (A)

$$=-\nu x^{-\nu-1}\left[\frac{x}{2\nu}\left\{J_{\nu-1}(x)+J_{\nu+1}(x)\right\}\right]+\frac{x^{-\nu}}{2}\left[J_{\nu-1}(x)-J_{\nu+1}(x)\right]$$

$$=-\frac{x^{-\nu}}{2}J_{\nu-1}(x)-\frac{x^{-\nu}}{2}J_{\nu+1}(x)+\frac{x^{-\nu}}{2}J_{\nu-1}(x)-\frac{x^{-\nu}}{2}J_{\nu+1}(x)$$

$$=-x^{-\nu}J_{\nu+1}(x)$$

$$=-x^{-\nu}J_{\nu+1}(x)$$

$$=R.H.S$$

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Example:- Prove that $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$

Solution:- As $J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\nu+n)}\left[\frac{x}{2}\right]^{2n+\nu}$

Put $\nu = 1/2$

$$J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma\left(1+\frac{1}{2}+n\right)}\left[\frac{x}{2}\right]^{2n+\frac{1}{2}}$$

$$J_{1/2}(x) = \frac{(-1)^0}{0!\,\Gamma\left(\frac{3}{2}+0\right)}\left[\frac{x}{2}\right]^{\frac{1}{2}} + \frac{(-1)^1}{1!\,\Gamma\left(\frac{3}{2}+1\right)}\left(\frac{x}{2}\right)^{2+\frac{1}{2}} + \frac{(-1)^2}{2!\,\Gamma\left(\frac{3}{2}+2\right)}\left(\frac{x}{2}\right)^{4+\frac{1}{2}} + \cdots$$

$$J_{1/2}(x) = \frac{1}{\frac{1}{2}\Gamma(1/2)} + \frac{(-1)}{\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)}\left(\frac{x}{2}\right)^{2+\frac{1}{2}} + \frac{1}{2\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)}\left(\frac{x}{2}\right)^{4+\frac{1}{2}} + \cdots$$

$$= \frac{1}{\frac{1}{2}\sqrt{\pi}}\left[\frac{x}{2}\right]^{1/2}\left[1 - \frac{(x/2)^2}{3/2} + \frac{(x/2)^4}{15/2} - \cdots\right]$$

$$= \frac{2}{\sqrt{\pi}}\cdot\frac{\sqrt{x}}{\sqrt{2}}\left[1 - \frac{x^2/4}{3/2} + \frac{x^4/16}{15/2} - \cdots\right]$$

$$= \frac{\sqrt{2}\sqrt{2}}{\sqrt{\pi}}\cdot\frac{\sqrt{x}}{\sqrt{2}}\left[1 - \frac{x^2}{(3)(2)} + \frac{x^4}{(15)(8)} - \cdots\right]$$

$$= \frac{\sqrt{2}\cdot\sqrt{x}}{\sqrt{\pi}}\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right]$$

Multiply and Divide by x

$$= \frac{\sqrt{2}\cdot\sqrt{x}}{\sqrt{\pi}}\cdot\frac{x}{x}\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right]$$

$$= \frac{\sqrt{2}\cdot}{\sqrt{\pi x}}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right] = \sqrt{\frac{2}{\pi x}}\sin x$$

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Example:- Prove that $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$

Solution:- As $J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\nu+n)} \left[\frac{x}{2}\right]^{2n+\nu}$

Put $\nu = -1/2$

$$J_{-1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1-\frac{1}{2}+n)} \left[\frac{x}{2}\right]^{2n-1/2}$$

$$= \frac{(-1)^0}{0!\,\Gamma(1/2)}\left(\frac{x}{2}\right)^{-1/2} + \frac{(-1)^1}{1!\,\Gamma(3/2)}\left(\frac{x}{2}\right)^{2-\frac{1}{2}} + \frac{(-1)^2}{2!\,\Gamma(5/2)}\left(\frac{x}{2}\right)^{4-\frac{1}{2}} + \cdots$$

$$= \frac{1}{\Gamma(1/2)}\left(\frac{x}{2}\right)^{-1/2} + \frac{(-1)}{\frac{1}{2}\Gamma(1/2)}\left(\frac{x}{2}\right)^{2-1/2} + \frac{1}{2\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)}\left(\frac{x}{2}\right)^{4-1/2} - \cdots$$

$$= \frac{1}{\Gamma(1/2)}\left(\frac{x}{2}\right)^{-1/2}\left[1 - \frac{(x/2)^2}{1/2} + \frac{(x/2)^4}{3/2} + \cdots\right]$$

$$= \frac{1}{\sqrt{\pi}}\cdot\left(\frac{2}{x}\right)^{1/2}\left[1 - \frac{x^2/4}{1/2} + \frac{x^4/16}{3/2} + \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[1 - \frac{x^2}{2} + \frac{x^4}{(3)(8)} - \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\cos x$$

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A/2013, A/2010

Example :- Show that $J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x} - \cos x\right]$

Solution :- Using the recurrence relation no (4)

$$J_{\nu+1}(x) = \frac{2\nu}{x} J_{\nu}(x) - J_{\nu-1}(x)$$

Put $\nu = \frac{1}{2}$

$$J_{\frac{1}{2}+1}(x) = \frac{2\cdot\frac{1}{2}}{x} J_{1/2}(x) - J_{\frac{1}{2}-1}(x)$$

$$J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x) \quad \text{—— (i)}$$

As we know that

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$$

and

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x \quad \text{—— (ii)}$$

Using these in (i)

$$J_{3/2}(x) = \frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{2}{\pi x}}\cos x$$

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x} - \cos x\right]$$

Hence prove

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A/2007

Example :- Show that $J_{-\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[-\frac{\cos x}{x} - \sin x\right]$

Solution :- Using the recurrence relation no (4)

$$J_{\nu+1}(x) = \frac{2\nu}{x} J_{\nu}(x) - J_{\nu-1}(x)$$

$$J_{\nu-1}(x) = \frac{2\nu}{x} J_{\nu}(x) - J_{\nu+1}(x)$$

put $\nu = -1/2$

$$J_{-\frac{1}{2}-1}(x) = \frac{2(-1/2)}{x} J_{-1/2}(x) - J_{-1/2+1}(x)$$

$$J_{-3/2}(x) = -\frac{1}{x} J_{-1/2}(x) - J_{1/2}(x) \qquad \text{---(i)}$$

AS we know that

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x \qquad \text{---(ii)}$$

Using these in (i)

$$J_{-\frac{3}{2}}(x) = -\frac{1}{x}\sqrt{\frac{2}{\pi x}} \cos x - \sqrt{\frac{2}{\pi x}} \sin x$$

$$J_{-\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{-\cos x}{x} - \sin x\right]$$

Hence proove

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Example :- Show that

$$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\left\{\frac{3-x^2}{x^2}\right\}\sin x - \frac{3\cos x}{x}\right]$$

Solution :- From recurrence relation (4) we have

$$J_{\nu+1}(x) \equiv \frac{2\nu}{x} J_{\nu}(x) - J_{\nu-1}(x) \quad \text{---(i)}$$

put $\nu = 3/2$

$$J_{\frac{3}{2}+1}(x) = \frac{2\cdot\left(\frac{3}{2}\right)}{x} J_{3/2}(x) - J_{\frac{3}{2}-1}(x)$$

$$J_{5/2}(x) = \frac{3}{x} J_{3/2}(x) - J_{1/2}(x) \quad \text{---(ii)}$$

put $\nu = 1/2$ in (i), we get $J_{3/2}$ then put in (ii)

$$J_{3/2}(x) = \frac{2(1/2)}{x}\left[J_{1/2}(x) - J_{\frac{1}{2}-1}(x)\right]$$

$$J_{3/2}(x) = \frac{1}{x}\left[J_{1/2}(x) - J_{-1/2}(x)\right]$$

put in (ii)

$$J_{5/2}(x) = \frac{3}{x}\left[\frac{1}{x}\left\{J_{1/2}(x) - J_{-1/2}(x)\right\}\right] - J_{1/2}(x)$$

$$= \frac{3}{x^2}\left[J_{\frac{1}{2}}(x)\right] - \frac{3}{x} J_{-1/2}(x) - J_{1/2}(x)$$

$$= \left(\frac{3}{x^2} - 1\right) J_{1/2}(x) - \frac{3}{x} J_{-1/2}(x)$$

$$= \left[\frac{3-x^2}{x^2}\right]\sqrt{\frac{2}{\pi x}}\sin x - \frac{3}{x}\sqrt{\frac{2}{\pi x}}\cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\left\{\frac{3-x^2}{x^2}\right\}\sin x - \frac{3}{x}\cos x\right]$$

Hence prove

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A/2015

Example :- Show that $J_{-\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3}{x}\sin x + \left(\frac{3}{x^2}-1\right)\cos x\right]$

Solution:- From recurrence relation (4), we have

$$J_{\nu+1}(x) = \frac{2\nu}{x} J_{\nu}(x) - J_{\nu-1}(x)$$

$$J_{\nu-1}(x) = \frac{2\nu}{x} J_{\nu}(x) - J_{\nu+1}(x) \quad \text{——(i)}$$

$\therefore$ Put $\nu = -3/2$

$$J_{-5/2}(x) = \frac{2(-3/2)}{x} J_{-3/2}(x) - J_{-\frac{3}{2}+1}(x)$$

$$J_{-5/2}(x) = -\frac{3}{x} J_{-3/2}(x) - J_{-1/2}(x) \quad \text{——(ii)}$$

Now put $\nu = -1/2$ in (i)

$$J_{-3/2}(x) = \frac{2(-1/2)}{x} J_{-1/2}(x) - J_{-\frac{1}{2}+1}(x)$$

$$J_{-3/2}(x) = -\frac{1}{x} J_{-1/2}(x) - J_{1/2}(x) \quad \text{——(iii)}$$

Put the value of (iii) in (ii)

$$J_{-5/2}(x) = -\frac{3}{x}\left[-\frac{1}{x} J_{-1/2}(x) - J_{1/2}(x)\right] - J_{-1/2}(x)$$

$$= \frac{3}{x^2} J_{-1/2}(x) + \frac{3}{x} J_{1/2}(x) - J_{-1/2}(x)$$

$$= \frac{3}{x} J_{1/2}(x) + \left(\frac{3}{x^2}-1\right) J_{-1/2}(x)$$

$$= \frac{3}{x}\sqrt{\frac{2}{\pi x}}\sin x + \left(\frac{3}{x^2}-1\right)\sqrt{\frac{2}{\pi x}}\cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{3}{x}\sin x + \left(\frac{3}{x^2}-1\right)\cos x\right]$$

Hence proove

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LEGENDRE'S EQUATION

The differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

is known as Legendre's equation. The above equation can also be written as

$$\frac{d}{dx}\left\{(1-x^2)\cdot\frac{dy}{dx}\right\} + n(n+1)y = 0$$

And its solution is called leger re function.

If n is positive integer, then the solution of Differential equation is called Legendre polynomial of Degree "n" and is denoted by $P_n(x)$.

P-U A/2011

Series Solution of Legendre D-E

As we know that

$$(1-x^2)y'' - 2xy' + \lambda y = 0 \quad \text{---(1)}$$

where $\lambda = n(n+1)$

Let $y = \sum_{m=0}^{\infty} C_m x^m$

$$y' = \sum_{m=1}^{\infty} C_m \cdot m x^{m-1}$$

$$y'' = \sum_{m=2}^{\infty} C_m \, m(m-1) x^{m-2}$$

Subsituting these in (1)

$$(1-x^2)\sum_{n=2}^{\infty} m(m-1)x^{m-2}\cdot C_m - 2x\sum_{m=1}^{\infty} m x^{m-1}\cdot C_m + \lambda\sum_{m=0}^{\infty} C_m x^m = 0$$

$$\sum_{m=2}^{\infty} m(m-1)x^{m-2}\cdot C_m - x^2\sum_{m=2}^{\infty} m(m-1)x^{m-2}\cdot C_m$$

$$-2x\sum_{m=1}^{\infty} m\, x^{m-1}\cdot C_m + \lambda\sum_{m=0}^{\infty} C_m x^m = 0$$

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$$\Rightarrow \sum_{m=0}^{\infty}(m+2)(m+1)C_{m+2}\cdot x^m - \sum_{m=2}^{\infty} m(m-1)C_m\cdot x^m$$

$$-2\sum_{m=1}^{\infty} m\cdot C_m\cdot x^m + \lambda\sum_{m=0}^{\infty} C_m x^m = 0$$

$$\Rightarrow (2)(1)C_1\cdot x^0 + (3)(2)C_3 x^1 + \sum_{m=2}^{\infty}(m+2)(m+1)C_{m+2}\, x^m$$

$$-\sum_{m=2}^{\infty} m(m-1)C_m\cdot x^m - 2C_1 x - 2\sum_{m=2}^{\infty} m C_m x^m$$

$$+\lambda[C_0 x^0 + C_1 x^1] + \lambda\sum_{m=2}^{\infty} C_m x^m = 0$$

$$\Rightarrow 2C_2 x^0 + \lambda C_0 x^0 + 6C_3 x - 2C_1 x + \lambda C_1 x + \sum_{m=2}^{\infty}\Big[(m+2)(m+1)C_{m+2}$$

$$+\{-m(m-1)-2m+\lambda\}C_m\Big]x^m = 0$$

$$\Rightarrow (2C_2+\lambda C_0)x^0 + (6C_3 - 2C_1 + \lambda C_1)x +$$

$$\sum_{n=2}^{\infty}\Big[(m+2)(m+1)C_{m+2} + \{-m^2+m-2m+n^2+n\}C_n\Big]x^m = 0$$

$$\Rightarrow (2C_2+\lambda C_0)x^0 + [6_3 + (\lambda-2)C_1]x + \sum_{m=2}^{\infty}(m+2)(m+1)C_{m+2}$$

$$+\{(n^2-m^2)+(n-m)\}C_m\Big]x^m = 0$$

Comparing the co-efficients

$$\Rightarrow 2C_2 + \lambda C_0 = 0 \qquad -\lambda C_0$$

$$\Longrightarrow \qquad C_2 = \frac{-\lambda C_0}{2}$$

$$6C_3 + (\lambda-2)C_1 = 0$$

$$C_3 = \frac{-(\lambda-2)C_1}{6} = \frac{(2-\lambda)C_1}{6}$$

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$$C_3 = \frac{(2-n^2+n)C_1}{6}$$

$$C_3 = \frac{-(n^2+n-2)C_1}{6}$$

$$C_3 = \frac{-(n^2+2n-n-2)C_1}{6}$$

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$$C_3 = \frac{-(n-1)(n+2)C_1}{6}$$

$$C_3 = \frac{-(n-1)(n+2)C_1}{3!}$$

Comparing the co-gficients of x^m, $m \geq 2$

$$(m+2)(m+1)C_{m+2} + [(n^2-m^2)+(n-m)]C_m = 0$$

$$\Rightarrow C_{m+2} = \frac{-[(n^2-m^2)+(n-m)]C_m}{(m+1)(m+2)}$$

$$\Rightarrow C_{m+2} = \frac{-[(n-m)(n+m)+(n-m)]C_m}{(m+1)(m+2)}$$

$$C_{m+2} = \frac{-[(n-m)(n+m+1)]C_m}{(m+1)(m+2)}$$

$$C_{m+2} = \frac{(m-n)(n+m+1)}{(m+1)(m+2)} C_m$$

put $m=2$

$$C_4 = \frac{(2-n)(n+3)}{(4)(3)} C_2$$

$$C_4 = \frac{(-1)(n-2)(n+3)}{(4)(3)} \cdot \frac{(-1)n(n+1)}{2} C_0$$

$$C_4 = \frac{(-1)^2 n(n-2)(n+1)(n+3)C_0}{4!}$$

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Put $m = 4$

$$C_6 = \frac{(-1)^3 n(n-2)(n-4)(n+1)(n+3)(n+5)}{6!} C_0$$

In general

$$C_{2K} = (-1)^K \frac{[n(n-2)\text{——}\{n-(2K-2)\}][(n+1)(n+3)\text{——}n+(2K-1)]}{(2K)!} C_0$$

For odd Co-efficients

put $m = 3$

$$C_5 = \frac{(3-n)(4+n)}{5\cdot 4} C_3$$

$$= \frac{(-1)(n-3)(n+4)}{5\cdot 4} \cdot \frac{(-1)(n-1)(n+2)}{3!} C_1$$

$$C_5 = \frac{(-1)^2 (n-3)(n-1)(n+2)(n+4)}{5!} C_1$$

$$C_7 = \frac{(5-n)(6+n)}{7\cdot 6} C_5$$

$$C_7 = \frac{-(n-5)(6+n)}{7\cdot 6} \cdot \frac{(-1)^2 (n-1)(n-3)(n+2)(n+4)}{5!} C_1$$

$$C_7 = \frac{(-1)^3 (n-1)(n-3)(n-5)(n+2)(n+4)(n+6)}{7!} C_1$$

In general

$$C_{2K+1} = \frac{(-1)^K [(n-1)(n-3)----(n-(2K-1))][(n+2)(n+4)\text{——}n+2K]}{(2K+1)!} C_1$$

As $y = \sum_{m=0}^{\infty} C_m x^m = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \text{——}$

$$= C_0 \left[1 - \frac{n(n+1)}{2!} x^2 + \frac{n(n-2)(n+1)(n+3)}{4!} x^4 \text{——}\right]$$

$$+ C_1 \left[x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!} x^5 \text{——}\right]$$

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Convergence of Legendre D-E

Using Absolute Ratio Test.

$$\lim_{m\to\infty}\left|\frac{u_{m+1}}{u_m}\right| = \lim_{m\to\infty}\left|\frac{c_{m+2}\,x^{m+2}}{c_m\,x^m}\right|$$

$$= \lim_{m\to\infty}\left|\frac{c_{m+2}}{c_m}\right|\left|x^2\right| \quad \text{---(1)}$$

As

$$c_{m+2} = \frac{(m-n)(m+n+1)}{(m+1)(m+2)}\,c_m$$

put in (1)

$$= \lim_{m\to\infty}\left|\frac{(m-n)(m+n+1)\,c_m}{(m+1)(m+2)\,c_m}\right|\left|x^2\right|$$

$$= \lim_{m\to\infty}\left|\frac{m(1-\frac{n}{m})\cdot m(1+\frac{n}{m}+\frac{1}{m})}{m(1+\frac{1}{m})\cdot m(1+\frac{2}{m})}\right|\left|x^2\right|$$

$$= \left|\frac{(1-\frac{n}{\infty})(1+\frac{n}{\infty}+\frac{1}{\infty})}{(1+\frac{1}{\infty})(1+\frac{2}{\infty})}\right|\left|x^2\right|$$

$$= \left|\frac{(1-0)(1+0+0)}{(1+0)(1+0)}\right|\left|x^2\right| \quad \ldots$$

$$= 1\cdot\left|x^2\right|$$

This series solution is convergent for

$$|x^2| < 1$$

$$|x| < 1$$

$$-1 < x < 1$$

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Question:- Show that $P_n(x)$ can be expressed as

$$P_n(x) = \sum_{K=0}^{N} C_{n-2K} \cdot x^{n-2K}$$

where $$C_{n-2K} = \frac{(-1)^K (2m-2K)!}{2^n K! (n-2K)! (n-K)!}$$

and

$$N = \left[\frac{n}{2}\right] = \begin{cases} n/2 & ; \text{if } n \text{ is even} \\ \frac{n-1}{2} & ; \text{if } n \text{ is odd} \end{cases}$$

Solution:- $P_n(x) = C_n x^n + C_{n-2} x^{n-2} + C_{n-4} x^{n-4} + \dots + C_0$

As $$C_m = \frac{(m+1)(m+2)}{(m-n)(m+n+1)} C_{m+2} \quad ; m \leq n-2$$

let $$C_n = \frac{(2n)!}{(n!)^2 2^n}$$

Let $m = n-2$

$$C_{n-2} = \frac{(n-1)n}{(-2)(2n-1)} C_n$$

$$= -\frac{(n-1)n}{2(2n-1)} \cdot \frac{(2n)!}{(n!)^2 2^n}$$

$$= -\frac{(n-1)n}{2(2n-1)} \cdot \frac{(2n)(2n-1)(2n-2)!}{2^n n(n-1)(n-2)! n(n-1)!}$$

$$= -\frac{(2n-2)!}{2^n (n-1)! (n-2)!}$$

Now put $m = n-4$

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$$C_{n-4} = \frac{(n-4+1)(n-4+2)}{(n-4-n)(n-4+n+1)} C_{n-4+2}$$

$$= \frac{(n-3)(n-2)}{(-4)(2n-3)} \cdot C_{n-2}$$

$$= \frac{(n-3)(n-2)}{-(4)(2n-3)} \cdot \frac{-(2n-2)!}{2^n (n-1)!(n-2)!}$$

$$= \frac{-(n-3)(n-2)}{(4)(2n-3)} \cdot \frac{-(2n-2)(2n-3)(2n-4)!}{2^n (n-1)(n-2)!(n-2)(n-3)(n-4)!}$$

$$= \frac{(-1)^2 \times (n-1)(2n-4)!}{2^2 \cdot 2^n (n-1)(n-2)!(n-4)!}$$

$$C_{n-4} = \frac{(-1)^2 (2n-4)!}{2!\, 2^n (n-2)(n-4)!}$$

In General

$$C_{n-2K} = \frac{(-1)^K (2m-2K)!}{2^n K!\, (n-2K)!\, (n-K)!}$$

which is also called Legendres Polynomial.

** **

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A/2012, A/2009

Generating function for Legendre Polynomial

Proof :-

For this we poove $(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n$

Consider $(1-2xt+t^2)^{-\frac{1}{2}}$ and expand by binomial Theorm

$$[1-t(2x-t)]^{-\frac{1}{2}} = 1+\left(\frac{-1}{2}\right)(-t(2x-t)) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}[-t(2x-t)]^2$$

$$+ \cdots + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\cdots\left[-\frac{1}{2}-(n-1)\right]}{n!}\cdot[-t(2x-t)]^n$$

$$= 1+\frac{t}{2}(2x-t) + \frac{1\cdot 3}{2^2\cdot 2!}t^2(2x-t)^2 + \cdots$$

$$+ \frac{1\cdot 3\cdot 5 \cdots 2n-3}{2^{n-1}(n-1)!}t^{n-1}(2x-t)^{n-1} + \frac{1\cdot 3\cdot 5 \cdots 2n-1}{2^n\cdot n!}t^n(2x-t)^n$$

Now comparing the co-efficient of t^n of R.H.S

$$= \frac{1\cdot 3\cdot 5 \cdots (2n-3)}{2^{n-1}(n-1)!}\left\{(n-1)(2x)^{n-2}(-1)\right\} + \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2^n\cdot n!}(2x)^n$$

$$= \frac{1\cdot 3\cdot 5 \cdots (2n-1)\,2^n\cdot x^n}{2^n\cdot n!} - \frac{(1\cdot 3\cdot 5 \cdots 2n-3)(n-1)\,2^{n-2}x^{n-2}}{2^{n-1}(n-1)!}$$

$$= \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{n!}x^n - \frac{1\cdot 3\cdot 5 \cdots (2n-3)(n-1)x^{n-2}}{2\cdot n!}\cdot\frac{n(2n-1)}{n(2n-1)}$$

$$= \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \cdots\right] \quad \text{--- (1)}$$

Now consider $P_n(x) = \sum_{K=0}^{n} \frac{(-1)^K (2n-2K)!}{2^n K!(n-2K)!(n-K)!}x^{n-2}$

Put $K=0$ we obtain the co-efficient of x^n i-e $\frac{(2n)!}{2^n(n!)^2}$

And put $K=1$ we get the co-efficient of x^{n-2}

i-e $\frac{(-1)(2n-2)!}{2^n(n-2)!(n-1)!}$

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Now from (1) Take the co-efficient of x^n

$$= \frac{1 \cdot 3 \cdot 5 \text{ —— } (2n-1)}{n!}$$

Multiply and Divide by $(2 \cdot 4 \cdot 6 \text{ —— } 2n)$

$$= \frac{1 \cdot 3 \cdot 5 \text{ —— } (2n-1)}{n!} \times \frac{2 \cdot 4 \cdot 6 \text{ —— } 2n}{2 \cdot 4 \cdot 6 \text{ —— } 2n}$$

$$= \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \text{ —— } (2n-1)(2n)}{n! \quad 2(1) \cdot 2(2) \cdot 2(3) \text{ —— } 2(n)}$$

$$= \frac{(2n)!}{n! \; 2^n [1 \cdot 2 \cdot 3 \cdot 4 \text{ —— } n]}$$

$$= \frac{(2n)!}{(n!)\, 2^n (n!)} = \frac{(2n)!}{2^n (n!)^2}$$

Now From (1) Take the co-efficent of x^{n-2}

$$= \frac{1 \cdot 3 \cdot 5 \text{ —— } (2n-1)}{n!} \left[\frac{-n(n-1)}{2(2n-1)} \right]$$

$$\therefore \frac{(2n)!}{2^n (n!)^2} \left[\frac{-n(n-1)}{2(2n-1)} \right]$$

$$= \frac{(2n)(2n-1)(2n-2)!\,[-n(n-1)]}{2^n [n(n-1)!] \cdot [n(n-1)(n-2)!]\, [2(2n-1)]}$$

$$= \frac{(-1)(2n-2)!}{2^n (n-1)!\,(n-2)!}$$

Similarly we compute x^{n-4}, x^{n-6} ——

And we get Legendre polynomial Terms :

Hence proved.

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A/2009

(i) Show that $P_n(1) = 1$

By using generating function

$$\sum_{n=0}^{\infty} P_n(x)\, t^n = \left[1 - 2xt + t^2\right]^{-\frac{1}{2}}$$

Put $x = 1$

$$\sum_{n=0}^{\infty} P_n(1)\, t^n = \left(1 - 2t + t^2\right)^{-\frac{1}{2}}$$

$$\therefore \sum_{n=0}^{\infty} P_n(1)\, t^n = (1-t)^{-1}$$

Expand by binomial Expansion

$$\sum_{n=0}^{\infty} P_n(1)\, t^n = 1 + t^2 + t^2 + t^3 + \text{——} t^n$$

By Equating co-efficient of t^n on both side

$$P_n(1) = 1$$

A/2009

(ii) Prove that $P_n(-x) = (-1)^n P_n(x)$

By using generating function

$$\sum_{n=0}^{\infty} P_n(x)\, t^n = \left(1 - 2xt + t^2\right)^{-\frac{1}{2}} \text{——} (1)$$

Now replacing "x" and "t" by "-x" and "-t"

$$\sum_{n=0}^{\infty} P_n(x)(-t)^n = \left(1 - 2xt + t^2\right)^{-\frac{1}{2}}$$

$$\sum_{n=0}^{\infty} (-1)^n P_n(-x)\, t^n = \left(1 - 2xt + t^2\right)^{-\frac{1}{2}} \text{——} (2)$$

Comparing (1) and (2)

$$P_n(x) = (-1)^n P_n(x) \qquad n = 0, 1, 2, 3 \text{——}$$

or

$$P_n(-x) = (-1)^n P_n(x) \qquad n = 0, 1, 2, 3 \text{——}$$

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(iii) Prove that $P_n(-1) = (-1)^n$

Take (ii) from Previous page

$$P_n(-x) = (-1)^n P_n(x)$$

put $x = 1$

$$P_n(-1) = (-1)^n P_n(1)$$

Since From (i) $P_n(1) = 1$

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$$P_n(-1) = (-1)^n \cdot 1$$

$P_n(-1) = (-1)^n$ Hence prove

* * * *

(iv) Prove that $P_n'(1) = \frac{1}{2} n(n+1)$

Solution :-

Legendre's equation is

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{---(1)}$$

If $P_n(x)$ is the solution of (1)

Then

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0$$

putting $x = 1$

$$(1-1^2)P_n''(1) - 2(1)P_n'(1) + n(n+1)P_n(1) = 0$$

$$(0)P_n''(1) - 2P_n'(1) + n(n+1)P_n(1) = 0$$

Since $P_n(1) = 1$

$$0 - 2P_n'(1) + n(n+1)\cdot 1 = 0$$

$$-2P_n'(1) = -n(n+1)$$

$P_n'(1) = \frac{1}{2} n(n+1)$ Hence prove

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(v) Prove that $P_n'(-1) = (-1)^{n-1} \frac{n}{2}(n+1)$

Solution:- Legendre's equation is

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{---(1)}$$

If $P_n(x)$ is the solution of (1)

Then $(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0$

Put $x = -1$

$$[1-(-1)^2]P_n''(-1) - 2(-1)P_n'(-1) + n(n+1)P_n(-1) = 0$$

As we know $P_n(-1) = (-1)^n$

Therefor $(0)P_n''(-1) + 2P_n'(-1) + n(n+1)(-1)^n = 0$

$$0 + 2P_n'(-1) - n(n+1)(-1)^{n-1} = 0$$

$$2P_n'(-1) = n(n+1)(-1)^{n-1}$$

$$P_n'(-1) = (-1)^{n-1} \cdot \frac{n(n+1)}{2}$$

** ═══ **

(vi) $P_{2n}(0) = (-1)^n \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots 2n}$ And $P_{2n+1}(0) = 0$

Solution:- As we know that

$$\sum t^{2n} P_{2n}(x) = [1-2xt+t^2]^{-\frac{1}{2}}$$

Putting $x = 0$

$$\sum t^{2n} P_{2n}(0) = [1-0+t^2]^{-\frac{1}{2}}$$

$$= [1+t^2]^{-1/2}$$

Expand by Binomial Theorem

$$\sum t^{2n} P_{2n}(0) = [1+t^2]^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)t^2 + \frac{\left[-\frac{1}{2}\right]\left[-\frac{1}{2}-1\right]}{2!}(t^2)^2$$

$$+ \frac{\left[-\frac{1}{2}\right]\left[-\frac{1}{2}-1\right]\left[-\frac{1}{2}-2\right]}{3!}(t^2)^3 + \cdots \frac{\left[-\frac{1}{2}\right]\left[-\frac{1}{2}-1\right]\cdots\left[-\frac{1}{2}-n+1\right]}{n!}(t^2)^n$$

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$$= 1 - \frac{t^2}{2} + \frac{\left[-\frac{1}{2}\right]\left[-\frac{3}{2}\right](t^2)^2}{2!} + \frac{\left[-\frac{1}{2}\right]\left[-\frac{3}{2}\right]\left[-\frac{5}{2}\right](t^2)^3}{3!}$$

$$\text{---} + \frac{\left[-\frac{1}{2}\right]\left[-\frac{3}{2}\right]\left[-\frac{5}{2}\right] \text{---} \left[-\frac{1}{2}-n+1\right](t^2)^n}{n!} + \text{---}$$

Equating the co-efficient of t^{2n} both sides we get

$$P_{2n}(e) = \frac{\left[-\frac{1}{2}\right]\left[-\frac{3}{2}\right]\left[-\frac{5}{2}\right] \text{---} \left[-\frac{1}{2}-n+1\right]}{n!}$$

$$= \frac{(-1)^n \; 1\cdot 3\cdot 5 \text{---} (2n-1)}{2^n \cdot n!}$$

$$= \frac{(-1)^n \; 1\cdot 3\cdot 5 \text{---} (2n-1)}{2^n \left[n\cdot(n-1)(n-2) \text{---} 3\cdot 2\cdot 1\right]}$$

$$= \frac{(-1)^n \cdot 1\cdot 3\cdot 5 \text{---} (2n-1)}{2(n)\cdot 2(n-1)\cdot 2(n-2) \text{---} 2(3)\cdot 2(2)\cdot 2(1)}$$

$$= \frac{(-1)^n \; 1\cdot 3\cdot 5 \text{---} (2n-1)}{2\cdot 4\cdot 6 \text{---} 2n}$$

Now Equating the co-efficient of t^{2n+1} of both side.

we get, $P_{2n+1}(0) = 0$

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A/2010

Question:- Prove that $\frac{1-z^2}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty}(2n+1)z^n P_n$

Solution:- As we know that

$$(1-2xz+z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} z^n P_n(x) \quad \text{---(1)}$$

Equation (1) Differentiate w.r.t "z", on both side

$$-\frac{1}{2}\left[1-2xz+z^2\right]^{\frac{-3}{2}}(-2x+2z) = \sum_{n=0}^{\infty} n z^{n-1} P_n(x)$$

$$\frac{(-\frac{1}{2})[-2(x-z)]}{\left[1-2xz+z^2\right]^{3/2}} = \sum_{n=0}^{\infty} n z^{n-1} P_n(x)$$

$$\frac{x-z}{\left[1-2xz+z^2\right]^{3/2}} = \sum_{n=0}^{\infty} n z^{n-1} P_n(x) \quad \text{---(2)}$$

Equation (2) multiply by 2z on both side

$$\frac{2xz-2z^2}{\left[1-2xz+z^2\right]^{3/2}} = \sum_{n=0}^{\infty} 2n z^n P_n(x) \quad \text{---(3)}$$

Adding (1) and (3), we get

$$(1-2xz+z^2)^{-\frac{1}{2}} + \frac{2xz-2z^2}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty} z^n P_n(x) + \sum_{n=0}^{\infty} 2n z^n P_n(x)$$

$$\frac{1-2xz+z^2+2xz-2z^2}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty}(2n+1)z^n P_n(x)$$

$$\boxed{\frac{1-z^2}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty}(2n+1)P_n(x)}$$

Hence prove

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Question:- Prove that

$$\frac{1+z}{z\sqrt{(1-2xz+z^2)}} - \frac{1}{z} = \sum_{n=0}^{\infty} (P_n + P_{n+1}) z^n$$

Solution:- Take R.H.S

$$\sum_{n=0}^{\infty} (P_n + P_{n+1}) z^n = \sum_{n=0}^{\infty} z^n P_n + \sum_{n=0}^{\infty} z^n P_{n+1}$$

$$= \sum_{n=0}^{\infty} z^n P_n + \frac{1}{z} \sum_{n=0}^{\infty} z^{n+1} P_{n+1} \quad \text{---(1)}$$

But $\sum_{n=0}^{\infty} z^n P_n = P_0 + zP_1 + z^2P_2 + z^3P_3 + \text{---}$

And $\sum_{n=0}^{\infty} z^{n+1} P_{n+1} = zP_1 + z^2P_2 + z^3P_3 + \text{---}$

Adding and Subtracting "P_0"

$$\sum_{n=0}^{\infty} z^{n+1} P_{n+1} = -P_0 + P_0 + zP_1 + z^2P_2 + z^3P_3 + \text{---}$$

$$= -P_0 + \sum z^n P_n$$

Putting the value of $\sum_{n=0}^{\infty} z^{n+1} P_{n+1}$ in (1), we get

$$\text{R.H.S} = \sum_{n=0}^{\infty} z^n P_n + \frac{1}{z}\left[\sum z^n P_n - P_0\right]$$

$$= \left[1 + \frac{1}{z}\right] \sum_{n=0}^{\infty} z^n P_n - \frac{P_0}{z}$$

$$= \left[\frac{1+z}{z}\right] (1-2xz+z^2)^{-\frac{1}{2}} - \frac{1}{z} \qquad (\because P_0 = 1)$$

$$= \frac{1+z}{z\sqrt{(1-2xz+z^2)}} - \frac{1}{z} = \text{L.H.S}$$

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Recurrence Formulae for Legendre Polynomials

(1) $(n+1)P_{n+1}(x) = (2n+1)x P_n(x) - n P_{n-1}(x)$

Proof :- As we know that

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$$

Differentiate w.r.t "t", we get

$$-\frac{1}{2}\left[1-2xt+t^2\right](-2x+2t) = \sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

$$\left[1-2xt+t^2\right]^{-3/2}(x-t) = \sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

Multiplying both side by $(1-2xt+t^2)$

$$\left[1-2xt+t^2\right]^{-\frac{1}{2}}(x-t) = (1-2xt+t^2)\sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

$$\sum_{n=0}^{\infty} t^n P_n\ [x-t] = \left[1-2xt+t^2\right]\sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

$$x\sum_{n=0}^{\infty} t^n P_n - \sum_{n=0}^{\infty} t^{n+1} P_n = \sum_{n=0}^{\infty} n t^{n-1} P_n(x) - 2x\sum_{n=0}^{\infty} n t^n P_n(x) + \sum_{n=0}^{\infty} n t^{n+1} P_n$$

$$x\sum_{n=0}^{\infty} t^n P_n(x) - \sum_{n=1}^{\infty} t^n P_{n-1}(x) = \sum_{n=0}^{\infty}(n+1) t^n P_{n+1}(x) - 2x\sum_{n=0}^{\infty} n t^n P_n(x) + \sum_{n=2}^{\infty}(n-1) P_{n-1}(x)\, t^n$$

Comparing the co-efficient of "t^n" on both side

$$x P_n(x) - P_{n-1}(x) = (n+1) P_{n+1}(x) - 2x\, n P_n(x) + (n-1) P_{n-1}(x)$$

$$x P_n(x) - P_{n-1}(x) + 2nx P_n(x) - (n-1) P_{n-1}(x) = (n+1) P_{n+1}(x)$$

$$\boxed{(n+1) P_n(x) = (2n+1) x P_n(x) - n P_{n-1}(x)}$$

Hence proved

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2) - $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$

Proof = As we know that

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x) \quad \text{---(1)}$$

Equation (1) Differentiate w.r.t "t"

$$-\frac{1}{2}[1-2xt+t^2]^{-3/2}(-2x+2t) = \sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

$$(x-t)(1-2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} n t^{n-1} P_n(x) \quad \text{---(2)}$$

Now Equation (1) Differentiate w.r.t "x"

$$-\frac{1}{2}\left[1-2xt+t^2\right]^{\frac{-3}{2}}(-2t) = \sum_{n=0}^{\infty} t^n P_n'(x)$$

$$t\left[1-2xt+t^2\right]^{\frac{-3}{2}} = \sum_{n=0}^{\infty} t^n P_n'(x) \quad \text{---(3)}$$

Equation (2) Dividing by (3)

$$\frac{(x-t)(1-2xt+t^2)^{-3/2}}{t(1-2xt+t^2)^{-3/2}} = \frac{\sum n t^{n-1} P_n(x)}{\sum t^n P_n'(x)}$$

$$(x-t)\sum_{n=0}^{\infty} t^n P_n'(x) = t\sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

$$x\sum_{n=0}^{\infty} t^n P_n'(x) - t\sum_{n=0}^{\infty} t^n P_n'(x) = \sum_{n=0}^{\infty} n t^n P_n(x)$$

$$x\sum_{n=0}^{\infty} t^n P_n'(x) - \sum_{n=0}^{\infty} t^{n+1} P_n'(x) = \sum_{n=0}^{\infty} n t^n P_n(x)$$

$$x\sum_{n=0}^{\infty} t^n P_n'(x) - \sum_{n=1}^{\infty} t^n P_{n-1}'(x) = \sum_{n=0}^{\infty} n t^n P_n(x)$$

Equating the Co-efficient of t^{n-1} on both side.

$$xP_n'(x) - P_{n-1}'(x) = nP_n(x)$$

$$\boxed{nP_n(x) = xP_n'(x) - P_{n-1}'(x)}$$

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(3)- $P'_{n+1} - P'_{n-1} = (2n+1)P_n$

Proof :- As we know from recurrence formula (1).

$$(n+1)P_{n+1} = (2n+1)xP_n - nP_{n-1}$$

Differentiate w.r.t "x"

$$(n+1)P'_{n+1} = (2n+1)[xP'_n + P_n] - nP'_{n-1}$$

$$(n+1)P'_{n+1} = (2n+1)xP'_n + (2n+1)P_n - nP'_{n-1} \quad \text{---(ii)}$$

As from recurrence formula (2)

$$nP_n = xP'_n - P'_{n-1}$$

$$nP_n + P'_{n-1} = xP'_n \quad \text{---(iii)}$$

Substituting the value of xP'_n from (iii) into (ii), we get-

$$(n+1)P'_{n+1} = (2n+1)[nP_n + P'_{n-1}] + (2n+1)P_n - nP'_{n-1}$$

$$(n+1)P'_{n+1} = (2n+1)nP_n + (2n+1)P'_{n-1} + (2n+1)P_n - nP'_{n-1}$$

$$(n+1)P'_{n+1} = (2n+1)(n+1)P_n + (2n+1-n)P'_{n-1}$$

$$(n+1)P'_{n+1} = (2n+1)(n+1)P_n + (n+1)P'_{n-1}$$

$$(n+1)P'_{n+1} = (n+1)[(2n+1)P_n + P'_{n-1}]$$

$$P'_{n+1} = (2n+1)P_n + P'_{n-1}$$

$$P'_{n+1} - P'_{n-1} = (2n+1)P_n$$

OR $\boxed{(2n+1)P_n = P'_{n+1} - P'_{n-1}}$

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(4) $P_n' - x P_{n-1}' = n P_{n-1}$

Proof:- As we know form Recurrence formula (1)

$$(n+1)P_{n+1} = (2n+1)x P_n - n P_{n-1}$$

Replacing "n" by (n-1), we get

$$nP_n = (2n-1)x P_{n-1} - (n-1) P_{n-2}$$

Differentiate the above, w.r.t "x"

$$nP_n' = (2n-1)\left[x P_{n-1}' + P_{n-1}\right] - (n-1) P_{n-2}'$$

$$nP_n' = (2n-1)x P_{n-1}' + (2n-1) P_{n-1} - (n-1) P_{n-2}'$$

$$nP_n' - (2n-1)x P_{n-1}' + (n-1)P_{n-2}' = (2n-1) P_{n-1}$$

$$nP_n' - \left[n+(n-1)\right]x P_{n-1}' + (n-1)P_{n-2}' = (2n-1) P_{n-1}$$

$$nP_n' - nx P_{n-1}' - (n-1)x P_{n-1}' + (n-1) P_{n-2}' = (2n-1) P_{n-1}$$

$$n\left[P_n' - x P_{n-1}'\right] - (n-1)\left[x P_{n-1}' - P_{n-2}'\right] = (2n-1) P_{n-1} \quad \text{— (A)}$$

As from Recurrence relation (2)

$$nP_n(x) = x P_n' - P_{n-1}'$$

Replace n by (n-1)

$$(n-1) P_{n-1}(x) = x P_{n-1}' - P_{n-2}' \quad \text{— (B)}$$

Substituting the values of (B) in (A)

$$n\left[P_n' - x P_{n-1}'\right] - (n-1)\left[(n-1) P_{n-1}\right] = (2n-1) P_{n-1}$$

$$n\left[P_n' - x P_{n-1}'\right] = (n-1)^2 P_{n-1} + (2n-1) P_{n-1}$$

$$= \left[(n-1)^2 + (2n-1)\right] P_{n-1}$$

$$= \left[n^2 + 1 - 2n + 2n - 1\right] P_{n-1}$$

$$n\left[P_n' - x P_{n-1}'\right] = n^2 P_{n-1}$$

$$\boxed{P_n' - x P_{n-1}' = n P_{n-1}}$$

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(5) $(x^2-1)P_n' = n[xP_n - P_{n-1}]$

Proof:- As we know from recurrence relation (4)

$$P_n' - xP_{n-1}' = nP_{n-1} \quad \text{---(i)}$$

And from recurrence formula (2)

$$xP_n' - P_{n-1}' = nP_n \quad \text{---(ii)}$$

Equation (ii) by "x"

$$x^2P_n' - xP_{n-1}' = nxP_n \quad \text{---(iii)}$$

Subtracting (iii) from (i)

$$P_n' - xP_{n-1}' = nP_{n-1}$$

$$x^2P_n' - xP_{n-1}' = nxP_n$$

$$(1-x^2)P_n' = n[P_{n-1} - xP_n]$$

OR $-(x^2-1)P_n' = n[P_{n-1} - xP_n]$

$$(x^2-1)P_n' = n[xP_n - P_{n-1}]$$

** **

(6) $(x^2-1)P_n' = (n+1)(P_{n+1} - xP_n)$

Proof:- As we know from recurrence formula (1)

$$(n+1)P_{n+1} = (2n+1)xP_n - nP_{n-1}$$

$$(n+1)P_{n+1} = [n+(n+1)]xP_n - nP_{n-1}$$

$$(n+1)P_{n+1} = nxP_n + (n+1)xP_n - nP_{n-1}$$

$$(n+1)P_{n+1} - (n+1)xP_n = nxP_n - nP_{n-1}$$

$$(n+1)[P_{n+1} - xP_n] = n[xP_n - P_{n-1}] \quad \text{---(i)}$$

As from recurrence formula (5)

$$(x^2-1)P_n' = n[xP_n - P_{n-1}] \quad \text{---(ii)}$$

From (i) & (ii)

$$\boxed{(x^2-1)P_n' = (n+1)[P_{n+1} - xP_n]}$$

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A/2016, A/2014

Question:- Prove the identity

$$x\frac{d}{dx}P_{K-1}(x)-\frac{d}{dx}P_{K-2}(x)=(K-1)P_{K-1}(x)$$

Proof:- We prove this identity by using generating function

$$(1-2xt+t^2)^{-\frac{1}{2}}=\sum_{K=0}^{\infty}t^K P_K(x) \quad \text{---(i)}$$

(i) Differentiate w.r.t "t"

$$-\frac{1}{2}\left[1-2xt+t^2\right]^{-3/2}(-2x+2t)=\sum_{K=0}^{\infty}Kt^{K-1}P_K(x)$$

$$(x-t)\left[1-2xt+t^2\right]^{-3/2}=\sum_{K=0}^{\infty}Kt^{K-1}P_K(x) \quad \text{---(ii)}$$

Now equation (i) Differentiate w.r.t "x"

$$-\frac{1}{2}\left[1-2xt+t^2\right]^{-3/2}(-2t)=\sum_{K=0}^{\infty}t^K P_K'(x)$$

$$t\left[1-2xt+t^2\right]^{-3/2}=\sum_{K=0}^{\infty}t^K P_K'(x) \quad \text{---(iii)}$$

Equation (ii) Dividing by (iii)

$$\frac{(x-t)\left[1-2xt+t^2\right]^{-3/2}}{t\left[1-2xt+t^2\right]^{-3/2}}=\frac{\sum Kt^{K-1}P_K(x)}{\sum t^K P_K'(x)}$$

$$\frac{x-t}{t}=\frac{\sum_{K=0}^{\infty}Kt^{K-1}P_K(x)}{\sum_{K=0}^{\infty}t^K P_K'(x)}$$

$$(x-t)\sum_{K=0}^{\infty}t^K P_K'(x)=t\sum_{K=0}^{\infty}Kt^{K-1}P_K(x)$$

$$x\sum_{K=0}^{\infty}t^K P_K'(x)-t\sum_{K=0}^{\infty}t^K P_K'(x)=\sum_{K=0}^{\infty}Kt^K P_K(x)$$

$$x\sum_{K=0}^{\infty}t^K P_K'(x)-\sum_{K=0}^{\infty}t^{K+1}P_K'(x)=\sum_{K=0}^{\infty}Kt^K P_K(x) \quad \text{---(iv)}$$

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$$\text{AS} \quad \sum_{K=0}^{\infty} t^{K+1} P_K'(x) = \sum_{K=1}^{\infty} t^K P_{K-1}'(x)$$

Substituting in (IV)

$$x\sum_{K=0}^{\infty} t^K P_K'(x) - \sum_{K=1}^{\infty} t^K P_{K-1}'(x) = \sum_{K=0}^{\infty} K t^K P_K(x)$$

Equating the Co-efficient of t^K on both side

$$x P_K'(x) - P_{K-1}'(x) = K P_K(x) \quad \text{--- (1)}$$

Replacing "K" by "K-1"

$$x P_{K-1}'(x) - P_{K-2}'(x) = (K-1) P_{K-1}(x)$$

OR

$$x \cdot \frac{d}{dx} P_{K-1}(x) - \frac{d}{dx} P_{K-2}(x) = (K-1) P_{K-1}(x)$$

A/2010, A/2007

RODRIGUE'S FORMULA

For Legendre's Polynomial

A/2010, A/2007

Question:- Show that $P_n(x) = \frac{1}{n! 2^n} \frac{d^n}{dx^n}(x^2-1)^n$

Proof:- Let $t(x) = (x^2-1)^n$

Differentiate w.r.t "x"

$$\frac{dt}{dx} = n(x^2-1)^{n-1}(2x)$$

$$\frac{dt}{dx} = 2nx(x^2-1)^n \cdot (x^2-1)^{-1}$$

$$(x^2-1)\frac{dt}{dx} = 2nx \cdot t(x)$$

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$$-[1-x^2]\frac{dt}{dx} - 2nx \cdot t(x) = 0$$

$$(1-x^2)\frac{dt}{dx} + 2nx \cdot t(x) = 0$$

Again differentiate w.r.t "x"

$$(1-x^2)\frac{d^2t}{dx^2} + \frac{dt}{dx}(-2x) + 2n\left[x \cdot \frac{dt}{dx} + t(x)\right] = 0$$

$$(1-x^2)\frac{d^2t}{dx^2} - 2x\frac{dt}{dx} + 2nx\frac{dt}{dx} + 2n \cdot t(x) = 0$$

$$(1-x^2)\frac{d^2t}{dx^2} - 2x(1-n)\frac{dt}{dx} + 2n \cdot t(x) = 0 \quad \text{---(i)}$$

Differentiating (i) expansion n-time using Leibniz formula

$$D^n[f \cdot g] = f \cdot g_n + n f_1 \cdot g_{n-1} + \frac{n(n-1)}{2!} f_2 \cdot g_{n-2} + \cdots + f_n \cdot g$$

$$\Rightarrow \left[(1-x^2)t_{n+2} - (n)2x\, t_{n+1} - \frac{2n(n-1)}{2!} t_n\right]$$

$$-2(1-n)\left[x t_{n+1} + n t_n\right] + 2n t_n = 0$$

$$\Rightarrow (1-x^2)t_{n+2} = 2nx - n(n-1)t_n - 2x(1-n)t_{n+1}$$

$$-2n(1-n)t_n + 2n t_n = 0$$

$$\Rightarrow (1-x^2)t_{n+2} + [-2nx - 2x(1-n)]t_{n+1}$$

$$+[-n(n-1) - 2n(1-n) + 2n]t_n = 0$$

$$\Rightarrow (1-x^2)t_{n+2} + [-2nx - 2x + 2nx]t_{n+1}$$

$$+[-n^2+n-2n+2n^2+2n]t_n = 0$$

$$\Rightarrow (1-x^2)t_{n+2} - 2x\, t_{n+1} + (n^2+n)t_n = 0$$

$$\Rightarrow (1-x^2)t_{n+2} - 2x\, t_{n+1} + n(n+1)t_n = 0$$

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$$(1-x^2)\frac{d^2}{dx^2}(t_n) - 2x\frac{d}{dx}(t_n) + n(n+1)t_n = 0$$

Since $t_n(x)$ satisfies the legendre D-E and $P_n(x)$ is also solution of legendre D-E. Therefore $P_n(x)$ and $t_n(x)$ are dependent functions

$$P_n(x) = C\, t_n(x)$$

where C is a constant to be determined

$$P_n(x) = C\frac{d^n}{dx^n}(x^2-1)^n \quad \text{(ii)}$$

$$C_n x^n + C_{n-2}x^{n-2} + C_{n-4}x^{n-4} + \cdots = C\frac{d}{dx^n}\left[x^{2n} - nx^{2n-1} + \cdots\right]$$

Equating terms with highest powers of x on both side

$$C_n = C\frac{d^n}{dx^n}[x^{2n}]$$

$$C_n = C\left[(2n)(2n-1)(2n-2)\cdots(2n-(n-1))\right]$$

$$\text{As } C_n = \frac{2n!}{(n!)^2 2^n}$$

$$\frac{2n!}{(n!)^2 2^n} = C\left[2n(2n-1)(2n-2)\cdots(n+1)\right]$$

$$= C\left[\frac{2n(2n-1)(2n-2)\cdots(n+1)}{n!} \times n!\right]$$

$$\frac{2n!}{(n!)^2 2^n} = \frac{C\,2n!}{n!} \Rightarrow C = \frac{1}{2^n \cdot n!}$$

putting the value of C in (ii)

$$\text{Hence } P_n(x) = \frac{1}{2^n \cdot n!}\frac{d^n}{dx^n}(x^2-1)^n$$

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A/2008

Question:- $\int_{-1}^{1} P_m(x) \cdot P_n(x) = \frac{2}{2n+1} \delta_{mn}$

$$\delta_{mn} = \begin{cases} 1 & \text{if } m \neq n \\ 0 & \text{if } m = n \end{cases}$$

To proove this we prove Legender polynomial is orthogonal if $m=n$ and orthonormal if $m \neq n$

Orthognality :- Consider legender D-E

$$(1-x^2)y'' - 2xy' + \lambda y = 0$$

OR

$$[(1-x^2)y']' + \lambda y = 0 \quad \text{——(i)}$$

Let λ_m and λ_n be the eigen values and P_m and P_n be the corresponding eigen solutions of legender's polynomial

Equation (i) Comparing with S-L equation

Then $p(x) = 1-x^2$, $q(x) = 0$ $\quad w(x) = 1$

$p(x) \to 0$ As $x \to \pm 1$

$$[(1-x^2)P_m']' + \lambda_m P_m = 0 \quad \text{——(ii)}$$

$$[(1-x^2)P_n']' + \lambda_n P_n = 0 \quad \text{——(iii)}$$

multiply equation (ii) by P_n and equation (iii) by P_m and then subtracting

$$[(1-x^2)P_m']' P_n + \lambda_m P_m P_n = 0$$

$$-[(1-x^2)P_n']' P_m + \lambda_n P_n P_m = 0$$

$$[(1-x^2)P_m']' P_n - [(1-x^2)P_n']' P_m + (\lambda_m - \lambda_n) P_m P_n = 0$$

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Integrating on both sides, we get

$$\int_{-1}^{1}[(1-x^2)P_m']'P_n\,dx - \int_{-1}^{1}[(1-x^2)P_n']'P_m\,dx =$$

$$(\lambda_n-\lambda_m)\int_{-1}^{1} P_n(x)P_m(x)\,dx$$

$$\Rightarrow P_n(x)(1-x^2)P_m'(x)\Big|_{-1}^{1} - \int_{-1}^{1}(1-x^2)P_m'(x)P_n'(x)\,dx$$

$$-P_m(x)(1-x^2)P_n'(x)\Big|_{-1}^{1} + \int_{-1}^{1}(1-x^2)P_n'(x)P_m'(x) = (\lambda_n-\lambda_m)\int_{-1}^{1} P_nP_m\,dx$$

$$\Rightarrow \Big[P_n(x)(1-x^2)P_m'(x)\Big]_{-1}^{1} - \Big[P_m(x)(1-x^2)P_n'(x)\Big]_{-1}^{1} = (\lambda_n-\lambda_m)\int_{-1}^{1} P_mP_n\,dx$$

$$P_n(1)(0)\cdot P_m'(1) - P_n(-1)(0)P_m'(-1) - P_m(1)(0)P_n'(1)$$

$$+ P_m(-1)(0)P_n'(-1) = (\lambda_n-\lambda_m)\int_{-1}^{1} P_n(x)P_m(x)\,dx$$

$$0 = (\lambda_n-\lambda_m)\int_{-1}^{1} P_n(x)P_m(x)\,dx$$

As $\lambda_n - \lambda_m \neq 0$ (∵ Because they Distinct)

So $\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0$

Hence they are orthogonal.

⟺

Now we prove the Normality Condition for Legendre polynomial.

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A/2015, A/2014

Normality Condition :- For Legender Polynomial

OR Show that $\int_{-1}^{1}[P_K(x)]^2 dx = \frac{2}{2K+1}$

Consider the generating function for Legender Polynomial

$$[1-2xz+z^2]^{-1/2} = \sum_{m=0}^{\infty} P_m(x) z^m$$

Squaring both side

$$(1-2xz+z^2)^{-1} = \left[\sum_{m=0}^{\infty} P_m(x)\, z^m\right]^2$$

$$\frac{1}{(1-2xz+z^2)} = \sum_{m=0}^{\infty}\sum_{m=0}^{\infty} [P_m(x)]^2 z^{2m}$$

Integrating on both sides

$$\int_{-1}^{1}\frac{1}{(1-2xz+z^2)}\cdot dx = \int_{-1}^{1}\sum_{m=0}^{\infty}[P_m(x)]^2 z^{2m}\cdot dx$$

$$-\frac{1}{2z}\int_{-1}^{1}\frac{-2z\cdot dx}{(1-2xz+z^2)} = \sum_{m=0}^{\infty}\int_{-1}^{1}[P_m(x)]^2 z^{2m}\cdot dx$$

$$-\frac{1}{2z}\left[\ln|1-2xz+z^2|\right]_{-1}^{1} = \sum_{m=0}^{\infty}\int_{-1}^{1}[P_m(x)]^2 z^{2m}\cdot dx$$

$$-\frac{1}{2z}\left[\ln(1-2z+z^2)-\ln(1+2z+z^2)\right] = \sum_{m=0}^{\infty}\int_{-1}^{1}[P_m(x)]^2 z^{2m}\,dx$$

$$\frac{-1}{2z}\left[\ln(1-z)^2-\ln(1+z)^2\right] = \sum_{m=0}^{\infty}\int_{-1}^{1}[P_m(x)]^2 z^{2m}\,dx$$

$$-\frac{1}{2z}\left[2\ln(1-z)-2\ln(1+z)\right] = \sum_{m=0}^{\infty}\int_{-1}^{1}[P_m(x)]^2 z^{2m}\,dx$$

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$$-\frac{1}{z}\left[\ln(1-z)-\ln(1+z)\right]=\sum_{m=0}^{\infty}\int_{-1}^{1}\left[P_m(x)\right]^2 z^{2m}\,dx$$

$$\frac{1}{z}\left[\ln(1+z)-\ln(1-z)\right]=\sum_{m=0}^{\infty}\int_{-1}^{1}\left[P_m(x)\right]^2 z^{2m}\,dx \quad \text{(A)}$$

As
$$\ln(1+\theta)=\theta-\frac{\theta^2}{2}+\frac{\theta^3}{3}-\frac{\theta^4}{4}+\text{——}$$
$$\ln(1-\theta)=-\theta-\frac{\theta^2}{2}-\frac{\theta^3}{3}-\frac{\theta^4}{4}-\text{——}$$

Using in (A)

$$\frac{1}{z}\left[\left\{z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}\text{——}\right\}-\left\{-z-\frac{z^2}{2}-\frac{z^3}{3}-\frac{z^4}{4}\text{——}\right\}\right]$$

$$=\sum_{m=0}^{\infty}\int_{-1}^{1}\left[P_m(x)\right]^2 z^{2m}\,dx$$

$$\frac{1}{z}\left[z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+\frac{z^5}{5}\text{——}+z+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\frac{z^5}{5}+\text{——}\right]$$

$$=\sum_{m=0}^{\infty}\int_{-1}^{1}\left[P_m(x)\right]^2 z^{2m}\,dx$$

$$\frac{1}{z}\left[2z+2\frac{z^3}{3}+2\frac{z^5}{5}+2\frac{z^7}{7}+\text{——}\right]=\sum_{m=0}^{\infty}\int_{-1}^{1}\left[P_m(x)\right]^2 z^{2m}\,dx$$

$$\frac{2z}{z}\left[1+\frac{z^2}{2}+\frac{z^4}{4}+\text{——}\right]=\sum_{m=0}^{\infty}\int_{-1}^{1}\left[P_m(x)\right]^2 z^{2m}\,dx$$

$$2\sum_{m=0}^{\infty}\frac{z^{2m}}{2}=\sum_{m=0}^{\infty}\int_{-1}^{1}\left[P_m(x)\right]^2 z^{2m}\,dx$$

Equating the co-efficient of z^{2m}

$$\int_{-1}^{1}\left[P_m(x)\right]^2\,dx=\frac{2}{2m+1}$$

Hence

$$\int_{-1}^{1}P_m(x)\,P_n(x)\,dx=\frac{2}{2m+1}\,\delta_{mn}$$

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HYPERGEOMETRIC FUNCTION

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Hypergeometric Series :-

The series of the form

$$1+\frac{\alpha\beta}{\gamma}\cdot x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\alpha+1)}\cdot\frac{x^2}{2!}+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}\cdot\frac{x^3}{3!}+\cdots+\frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+m-1)\beta(\beta+1)\cdots(\beta+m-1)}{\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+m-1)}\cdot\frac{x^m}{m!}$$

$$F_{21}(\alpha,\beta,\gamma;x)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\cdot\frac{x^n}{n!}\quad\text{---(1)}$$

where

$$(\alpha)_m=(\alpha)(\alpha+1)(\alpha+2)\cdots[\alpha+(m-1)]$$

$$(\beta)_m=(\beta)(\beta+1)(\beta+2)\cdots[\beta+(m-1)]$$

$$(\gamma)_m=(\gamma)(\gamma+1)(\gamma+2)\cdots[\gamma+(m-1)]$$

where 21 means two parameters in Numerator and one parameter in denominator.

Also Note that $\gamma\neq 0$ otherwise function undefined

The series (1) is called Hypergeometric Series.

The Pochhammer Symbol

The symbol $(\alpha)_n$ defined by

$$(\alpha)_0=1$$

$$(\alpha)_1=\alpha$$

$$(\alpha)_2=\alpha(\alpha+1)$$

$$(\alpha)_3=\alpha(\alpha+1)(\alpha+2)$$

$$(\alpha)_n=\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)$$

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A/2008 (40)

HYPERGEOMETRIC D-E

The Differential equation of the form

$$x(1-x)y''(x) + [\gamma - (\alpha+\beta+1)x]y' - \alpha\beta y = 0$$

Series Solution of HYPEGEOMETRIC D-E

We use Frobenious method to find Series solution of Hypergeometric D-E.

Also Here $x = 0$ is a regular singular point.

Let $y(x) = \sum_{n=0}^{\infty} C_n x^{n+r}$

$$y'(x) = \sum_{n=0}^{\infty} C_n (n+r) x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{\infty} C_n (n+r)(n+r-1) x^{n+r-2}$$

Substitating these in Hypergeometric D-E

$$x(1-x)\left[\sum_{n=0}^{\infty} C_n (n+r)(n+r-1) x^{n+r-2}\right] + [\gamma - (\alpha+\beta+1)x]$$

$$\sum_{n=0}^{\infty} (n+r) x^{n+r-1} \cdot C_n - \alpha\beta \sum_{n=0}^{\infty} C_n x^{n+r} = 0$$

$$x\sum_{n=0}^{\infty} C_n (n+r)(n+r-1) x^{n+r-2} - x^2 \sum_{n=0}^{\infty} C_n (n+r)(n+r-1) x^{n+r-2}$$

$$+ \left[\gamma \sum_{n=0}^{\infty} (n+r) x^{n+r-1} \cdot C_n - (\alpha+\beta+1)x \sum_{n=0}^{\infty} (n+r) x^{n+r-1} \cdot C_n\right]$$

$$- \alpha\beta \sum_{n=0}^{\infty} C_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} C_n (n+r)(n+r-1) x^{n+r-1} - \sum_{n=0}^{\infty} C_n (n+r)(n+r-1) x^{n+r}$$

$$+ \gamma \sum_{n=0}^{\infty} (n+r) x^{n+r-1} \cdot C_n - (\alpha+\beta+1) \sum_{n=0}^{\infty} (n+r) x^{n+r} \cdot C_n - \alpha\beta \sum_{n=0}^{\infty} C_n x^{n+r} = 0$$

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$$\gamma(\gamma-1)C_0 x^{\gamma-1} + \sum_{n=1}^{\infty}(n+\gamma)(n+\gamma-1)C_n x^{n+\gamma-1} - \sum_{n=0}^{\infty}(n+\gamma)(n+\gamma-1)C_n x^{n+\gamma}$$

$$+\gamma(\gamma)C_0 x^{\gamma-1} + \gamma\sum_{n=1}^{\infty}(n+\gamma)x^{n+\gamma-1}C_n - (\alpha+\beta+1)\sum_{n=0}^{\infty}C_n(n+\gamma)x^{n+\gamma}$$

$$-\alpha\beta\sum_{n=0}^{\infty}C_n x^{n+\gamma} = 0$$

As $\sum_{n=1}^{\infty}(n+\gamma)(n+\gamma-1)C_n x^{n+\gamma-1} = \sum_{n=0}^{\infty}(n+\gamma+1)(n+\gamma)C_{n+1}x^{n+\gamma}$

And $\sum_{n=1}^{\infty}(n+\gamma)x^{n+\gamma-1}C_n = \sum_{n=0}^{\infty}(n+\gamma+1)C_{n+1}x^{n+\gamma}$

$$\gamma(\gamma-1)C_0 x^{\gamma-1} + \sum_{n=0}^{\infty}(n+\gamma+1)(n+\gamma)C_{n+1}x^{n+\gamma} - \sum_{n=0}^{\infty}(n+\gamma)(n+\gamma-1)C_n x^{n+\gamma}$$

$$+\gamma(\gamma)C_0 x^{\gamma-1} + \gamma\sum_{n=0}^{\infty}(n+\gamma+1)C_{n+1}x^{n+\gamma} - (\alpha+\beta+1)\sum_{n=0}^{\infty}C_n(n+\gamma)x^{n+\gamma}$$

$$-\alpha\beta\sum_{n=0}^{\infty}C_n x^{n+\gamma} = 0$$

$$\left[\gamma(\gamma-1)C_0 x^{\gamma-1} + \gamma(\gamma)C_0 x^{\gamma-1}\right] + \sum_{n=0}^{\infty}\left[(n+\gamma+1)(n+\gamma)C_{n+1}\right.$$

$$-(n+\gamma)(n+\gamma-1)C_n + \gamma(n+\gamma+1)C_{n+1} - (\alpha+\beta+1)C_n(n+\gamma)$$

$$\left.-\alpha\beta(C_n)\right]x^{n+\gamma} = 0$$

$$\Rightarrow \left[\gamma(\gamma-1)+\gamma(\gamma)\right]C_0 x^{\gamma-1} + \sum_{n=0}^{\infty}\left[(n+\gamma+1)\{n+\gamma+\gamma\}C_{n+1}\right.$$

$$\left.-\left[(n+\gamma)\{(n+\gamma-1)+(\alpha+\beta+\ )\}+\alpha\beta\right]C_n = 0\right.$$

Comparing the Co-efficent of $x^{\gamma-1}$

$$\left[\gamma(\gamma-1)+\gamma(\gamma)\right]C_0 = 0$$

$$\gamma\left[\gamma-1+\gamma\right] = 0 \quad \because C_0 \neq 0$$

$\gamma = 0$ or $\gamma-1+\gamma = 0$

$\gamma = 1-\gamma$

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Comparing the co-efficient of $x^{n+\gamma}$

$$(n+\gamma+1)(n+\gamma+\gamma)C_{n+1} - \{(n+\gamma)[(n+\gamma-1)+(\alpha+\beta+1)] + \alpha\beta\}C_n = 0$$

$$C_{n+1} = \frac{[\{(n+\gamma)(n+\gamma+\alpha+\beta)\}+\alpha\beta]}{(n+\gamma+1)(n+\gamma+\gamma)} C_n \quad ; n \geq 0$$

For $\gamma = 0$

$$C_{n+1} = \frac{[n(n+\alpha+\beta)+\alpha\beta]}{(n+1)(n+\gamma)} C_n$$

$$" = \frac{[n^2+n\alpha+n\beta+\alpha\beta]}{(n+1)(n+\gamma)} C_n$$

$$" = \frac{[n(n+\alpha)+\beta(n+\alpha)]}{(n+1)(n+\gamma)} C_n$$

$$C_{n+1} = \frac{[(n+\alpha)(n+\beta)]}{(n+1)(n+\gamma)} C_n$$

For $n=0$

$$C_1 = \frac{\alpha\beta}{\gamma} C_0$$

Let $(C_0=1) \Rightarrow C_1 = \frac{\alpha\beta}{\gamma}$

For $n=1$

$$C_2 = \frac{(1+\alpha)(1+\beta)}{(2)(1+\gamma)} C_1$$

$$C_2 = \frac{(1+\alpha)(1+\beta)}{(2)(1+\gamma)} \cdot \frac{\alpha\beta}{\gamma}$$

$$C_2 = \frac{\alpha(1+\alpha)\beta(1+\beta)}{\gamma(1+\gamma)\ 2!}$$

Continuing this process we get

$$C_n = \frac{[\alpha(\alpha+1) \text{ --- } \alpha+(n-1)][\beta(\beta+1) \text{ --- } \beta+(n-1)]}{[\gamma(\gamma+1)(\gamma+2) \text{ --- } \gamma+(n-1)]\ n!}$$

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$$C_n = \frac{(\alpha)_n (\beta)_n}{(\gamma)_n \cdot n!}$$

As $y(x) = \sum_{n=0}^{\infty} C_n x^{n+\gamma}$

For $\gamma = 0$

$$y(x) = \sum_{n=0}^{\infty} C_n x^n$$

$$y(x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} x^n$$

When $\gamma = 1 - \gamma$

$$C_{n+1} = \frac{[(n+\gamma)(n+\gamma+\alpha+\beta) + \alpha\beta]\, C_n}{(n+\gamma+1)(n+\gamma+\gamma)}$$

$$C_{n+1} = \frac{[(n+1-\gamma)(n+1-\gamma+\alpha+\beta) + \alpha\beta]\, C_n}{(n+1-\gamma+1)(n+1-\gamma+\gamma)}$$

$$C_{n+1} = \frac{[(n+1-\gamma)(n+1-\gamma+\alpha) + (n+1-\gamma)(\beta) + \alpha\beta]\, C_n}{(n+2-\gamma)(n+1)}$$

$$C_{n+1} = \frac{[(n+1-\gamma)(n+1-\gamma+\alpha) + \beta(n+1-\gamma+\alpha)]\, C_n}{(n+2-\gamma)(n+1)}$$

$$C_{n+1} = \frac{(n+1-\gamma+\alpha)(n+1-\gamma+\beta)\, C_n}{(n+2-\gamma)(n+1)}$$

put $n=0 \Rightarrow C_1 = \frac{(\alpha-\gamma+1)(\beta-\gamma+1)\, C_0}{(2-\gamma)(1)}$

let $(C_0 = 1) \Rightarrow C_1 = \frac{(\alpha-\gamma+1)(\beta-\gamma+1)}{(2-\gamma)(1)}$

put $n=1 \Rightarrow C_2 = \frac{(2+\alpha-\gamma)(2+\beta-\gamma)\, C_1}{(3-\gamma)\, 2}$

$$C_2 = \frac{(2+\alpha-\gamma)(2+\beta-\gamma)(\alpha-\gamma+1)(\beta-\gamma+1)}{2!\,(2-\gamma)(3-\gamma)}$$

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Similarly

$$C_n = \frac{(\alpha-\gamma+1)(\alpha-\gamma+2)\cdots(\alpha-\gamma+n)(\beta-\gamma+n)(\beta-\gamma+2)\cdots(\beta-\gamma+n)}{n!\,(2-\gamma)(3-\gamma)\cdots(\overline{n+1}-\gamma)}$$

$$C_n = \frac{(\alpha-\gamma+1)_n\ (\beta-\gamma+1)_n}{n!\,(2-\gamma)_n}$$

As $y(x) = \sum_{n=0}^{\infty} C_n x^{n+\gamma}$

For $\gamma = 1-\gamma \Rightarrow y(x) = \sum_{n=0}^{\infty} C_n x^{n+1-\gamma}$

$$= x^{1-\gamma} \sum_{n=0}^{\infty} C_n x^n$$

$$y(x) = x^{1-\gamma} \sum_{n=0}^{\infty} \frac{(\alpha-\gamma+1)_n(\beta-\gamma+1)_n}{(2-\gamma)_n\ n!}\cdot x^n$$

Convergence of Series Solution

$$F(\alpha,\beta,\gamma;x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} x^n$$

By Ratio Test

$$\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\frac{(\alpha)_{n+1}(\beta)_{n+1}\cdot x^{n+1}}{(\gamma)_{n+1}(n+1)!} \times \frac{n!\,(\gamma)_n}{(\alpha)_n(\beta)_n x^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{(\alpha)(\alpha+1)(\alpha+2)\cdots(\alpha+n)\,\beta(\beta+1)\cdots(\beta+n)\,\gamma(\gamma+1)\cdots(\gamma+\overline{n-1})\,n!\,x}{\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n)\,\alpha(\alpha+1)\cdots(\alpha+\overline{n-1})\,\beta(\beta+1)\cdots(\beta+\overline{n-1})\,(n+1)n!}\right|$$

$$\lim_{n\to\infty}\left|\frac{(\alpha+n)(\beta+n)}{(\gamma+n)(n+1)}\right||x|$$

$$\lim_{n\to\infty}\left|\frac{n\left(\frac{\alpha}{n}+1\right)n\left(\frac{\beta}{n}+1\right)}{n\left(\frac{\gamma}{n}+1\right)n\left(1+\frac{1}{n}\right)}\right||x|$$

$$\left|\frac{\left(\frac{\alpha}{\infty}+1\right)\left(\frac{\beta}{\infty}+1\right)}{\left(\frac{\gamma}{\infty}+1\right)\left(1+\frac{1}{\infty}\right)}\right||x|$$

$$\frac{(0+1)(0+1)}{(0+1)(1+0)}|x|$$

$$|x| < 1$$

Series is converges for $|x|<1$

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Some Important Results

(1) = Gamma function = $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx = (n-1)!$

$$\Gamma(n) = \lim_{x \to \infty} \frac{(x-1)!}{(n)_x} x^n$$

(2) = Beta Function

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \cdot dx$$

(3) = $B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

(4) = $\Gamma(n+1) = n\Gamma(n)$

(5) = $(1-x)^{-\alpha} = \sum_{n=0}^{\infty} (\alpha)_n \dfrac{x^n}{n!}$

Proof $(1-x)^{-\alpha} = 1 + \alpha x + \dfrac{\alpha(\alpha+1)}{2!} x^2 + \dfrac{\alpha(\alpha+1)(\alpha+2)}{3!} x^3$

$$+ \cdots + \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)}{n!} x^n$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha)_n x^n}{n!}$$

(6) $\Rightarrow \dfrac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = (\alpha)_n$

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Proof $\dfrac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \Gamma[(\alpha+n-1)+1] / \Gamma(\alpha)$

using $\Gamma(n+1) = n\Gamma(n)$

$$= \frac{(\alpha+n-1)\Gamma(\alpha+n-1)}{\Gamma(\alpha)}$$

$$= \frac{(\alpha+n-1)(\alpha+n-2)\cdots(\alpha)\Gamma(\alpha)}{\Gamma(\alpha)}$$

$$= \alpha(\alpha+1)\cdots(\alpha+n-2)(\alpha+n-1)$$

$$= (\alpha)_n$$

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A/2016, A/2009

Integral Representation of Hypergeometric Function

Prove $F_{21}(\alpha,\beta,\gamma;z)=\dfrac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-tz)^{-\beta}\,dt$

Proof :- Consider the Beta function defined by

$$B(\alpha+n,\gamma-\alpha)=\frac{\Gamma(\alpha+n)\Gamma(\gamma-\alpha)}{\Gamma(\alpha+n+\gamma-\alpha)}$$

$$=\frac{\Gamma(\alpha+n)\Gamma(\gamma-\alpha)}{\Gamma(n+\gamma)}$$

Multiply And Divid by $\Gamma(\gamma)\Gamma(\alpha)$

$$=\frac{\Gamma(\alpha+n)\Gamma(\gamma-\alpha)}{\Gamma(n+\gamma)}\times\frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma)}$$

$$=\left[\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\right]\left[\frac{\Gamma(\gamma)}{\Gamma(\gamma+n)}\right]\left[\frac{\Gamma(\gamma-\alpha)\Gamma(\alpha)}{\Gamma(\gamma)}\right]$$

$$B(\alpha+n,\gamma-\alpha)=\frac{(\alpha)_n}{(\gamma)_n}\cdot\frac{\Gamma(\gamma-\alpha)\Gamma(\alpha)}{\Gamma(\gamma)}\quad\text{---(1)}$$

$$\frac{(\alpha)_n}{(\gamma)_n}=\frac{\Gamma(\gamma)\,B(\alpha+n,\gamma-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\alpha)}$$

$$\frac{(\alpha)_n}{(\gamma)_n}=\frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\alpha)}\int_0^1 t^{\alpha+n-1}(1-t)^{\gamma-\alpha-1}\,dt$$

$$\frac{(\alpha)_n(\beta)_n}{(\gamma)_n\,n!}z^n=\frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\alpha)}\int_0^1 t^{\alpha+n-1}(1-t)^{\gamma-\alpha-1}\,dt\cdot\frac{(\beta)_n z^n}{n!}$$

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$$\sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n\, n!} z^n = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1} \sum_{n=0}^{\infty} \frac{(\beta)_n (tz)^n}{n!}\, dt$$

As By Result No(5)

$$= \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-tz)^{-\beta}\, dt$$

$$F_{21}(\alpha,\beta,\gamma,z) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-tz)^{-\beta}\, dt$$

The above Integral representation as Shown by Euler in 1748.

At $z=1$

$$F_{21}(\alpha,\beta,\gamma,1) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-t)^{-\beta}\, dt$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1-\beta}\, dt$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} B(\alpha, \gamma-\alpha-\beta)$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \cdot \frac{\Gamma(\alpha)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\alpha+\gamma-\alpha-\beta)}$$

$$\because B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$F_{21}(\alpha,\beta,\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}$$

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A/2015

Question:- Show that for Hypergeometric function

$$F\left[-\frac{n}{2}; -\frac{n}{2}+\frac{1}{2}, \beta+\frac{1}{2}, 1\right] = \frac{2^n(\beta)_n}{(2\beta)_n}$$

Solution:-

As $F(\alpha,\beta,\gamma;1) = \dfrac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}$

, is real if $\gamma-\alpha-\beta>0$

$\therefore$ and $(\gamma)>0$ $\therefore$ $\gamma>\alpha+\beta$

So $\beta+\frac{1}{2} > -\frac{n}{2}-\frac{n}{2}+\frac{1}{2}$

$= -n+\frac{1}{2}$

$\beta > -n$ Hence the condition of validity is true.

Now $F\left[-\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}, \beta+\frac{1}{2}; 1\right]$

$$= \frac{\Gamma(\beta+\frac{1}{2})\Gamma(\beta+\frac{1}{2}+\frac{n}{2}+\frac{n}{2}-\frac{1}{2})}{\Gamma(\beta+\frac{1}{2}+\frac{n}{2})\Gamma(\beta+\frac{1}{2}+\frac{n}{2}-\frac{1}{2})}$$

$$= \frac{\Gamma(\beta+\frac{1}{2})\,\Gamma(\beta+n)}{\Gamma(\beta+\frac{1}{2}+\frac{n}{2})\Gamma(\beta+\frac{n}{2})} \times \frac{\Gamma(\beta)}{\Gamma(\beta)}$$

$$= \frac{\Gamma(\beta)\Gamma(\beta+\frac{1}{2})}{\Gamma(\beta+\frac{n}{2})\Gamma(\beta+\frac{n}{2}+\frac{1}{2})} \times \frac{\Gamma(\beta+n)}{\Gamma(\beta)}$$

Using Dublication Formula

$$\Gamma(z)\Gamma(z+1/2) = \sqrt{\pi}\,\Gamma 2z \quad 2^{1-2z}$$

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$$= \frac{\sqrt{\pi}\,\Gamma(2\beta)\;2^{1-2\beta}}{\sqrt{\pi}\,\Gamma 2(\beta+\frac{n}{2})\cdot 2^{1-2(\beta+n/2)}}(\beta)_n$$

$$= \frac{\Gamma(2\beta)\;2^{1-2\beta}}{\Gamma(2\beta+n)\;2^{1-2\beta-n}}\cdot(\beta)_n$$

using $\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = (\alpha)_n$

$$= (2\beta)_n \frac{2^{1-2\beta}}{2^{1-2\beta}\cdot 2^{-n}}(\beta)_n$$

$$= \frac{2^n(\beta)_n}{(2\beta)_n}$$ Hence Prove

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A/2012 , A/2009

QUESTION :- Show that

$$\frac{d}{dx}F(\alpha,\beta,\gamma;x) = \frac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1,\gamma+1;x)$$

Solution :- Take L.H.S $= \frac{d}{dx}[F(\alpha,\beta,\gamma,x)]$

$$= \frac{d}{dx}\left[\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n\cdot n!}x^n\right]$$

$$= \frac{d}{dx}\left[1+\frac{\alpha\beta}{\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!}+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{x^3}{3!}+\cdots\right]$$

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$$\frac{\alpha\beta}{\gamma} + \frac{\alpha(\alpha+1)\,\beta(\beta+1)}{\gamma(\gamma+1)}\frac{2x}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)\,\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{3x^2}{3!} + \cdots$$

$$\frac{\alpha\beta}{\gamma}\left[1 + \frac{(\alpha+1)(\beta+1)}{(\gamma+1)}x + \frac{(\alpha+1)(\alpha+2)\,(\beta+1)(\beta+2)}{(\gamma+1)(\gamma+2)}\frac{x^2}{2!} - \right.$$

$$\frac{\alpha\beta}{\gamma}\left[\sum_{n=0}^{\infty}\frac{(\alpha+1)_n\,(\beta+1)_n}{(\gamma+1)_n}x^n\right]$$

$$\frac{\alpha\beta}{\gamma}F(\alpha+1,\ \beta+1,\ \gamma+1,\ 1) \quad \text{proved}$$

A/2013

Question:- Show that

$$\frac{d^2}{dx^2}F(a,b,c;x) = \frac{a(a+1)\,b(b+1)}{c(c+1)}F[a+2,\ b+2,\ c+2,\ x]$$

Solution:- As

$$F(a,b,c;x) = \left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\cdot\frac{x^n}{n!}\right]$$

$$F(a,b,c;x) = \left[1 + \frac{ab}{c}x + \frac{a(a+1)\,b(b+1)}{c(c+1)}\frac{x^2}{2!} + \right.$$

$$\left.\frac{a(a+1)(a+2)\,b(b+1)(b+2)}{c(c+1)(c+2)}\frac{x^3}{3!} + \cdots\right]$$

Diff w.r.t "x"

$$\frac{d}{dx}F(a,b,c;x) = \left[0 + \frac{ab}{c} + \frac{a(a+1)\,b(b+1)}{c(c+1)}\frac{2x}{2!} + \right.$$

$$\left. + \frac{a(a+1)(a+2)\,b(b+1)(b+2)}{c(c+1)(c+2)}\cdot\frac{3x^2}{6} + \cdots\right.$$

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Again Diff w.r.t "x"

$$\frac{d^2}{dx^2}F(a,b,c;x) = \left[0 + \frac{a(a+1)b(b+1)}{c(c+1)} + \frac{a(a+1)(a+2)\,b(b+1)(b+2)}{c(c+1)(c+2)}x + \cdots\right.$$

$$= \frac{a(a+1)b(b+1)}{c(c+1)}\left[1 + \frac{(a+2)(b+2)}{(c+2)}x + \frac{(a+2)(a+3)(b+2)(b+3)}{(c+2)(c+3)}\frac{x^2}{2!} + \cdots\right.$$

$$= \frac{a(a+1)b(b+1)}{c(c+1)}\left[\sum_{n=0}^{\infty}\frac{(a+2)_n\,(b+2)_n}{(c+2)_n}\cdot\frac{x^n}{n!}\right]$$

$$= \frac{a(a+1)b(b+1)}{c(c+1)}F(a+2, b+2, c+2; x)$$

A/2008

Question:- Prove that:-

$$\frac{d^p}{dx^p}\,{}_1F_1(\alpha,\beta;x) = \frac{(\alpha)_p}{(\beta)_p}\,{}_1F_1(\alpha+p,\beta+p,x)$$

Solution:- As

$${}_1F_1(\alpha,\beta,x) = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\beta)_n}\frac{x^n}{n!}$$

$${}_1F_1(\alpha,\beta,x) = 1 + \frac{\alpha}{\beta}x + \frac{\alpha(\alpha+1)}{\beta(\beta+1)}\frac{x^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)}\frac{x^2}{2!}$$

Diff w.r.t "x"

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$$\frac{d}{dx}\left[{}_1F_1(\alpha,\beta,x)\right] = 0 + \frac{\alpha}{\beta} + \frac{\alpha(\alpha+1)}{\beta(\beta+1)}\frac{x}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)}\frac{x^2}{3!}$$

Again Diff w.r.t "x"

$$\frac{d^2}{dx^2}\left[{}_1F_1(\alpha,\beta;x)\right] = 0 + \frac{\alpha(\alpha+1)}{\beta(\beta+1)} + \frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)}\frac{x}{2!}$$

$$\frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}{\beta(\beta+1)(\beta+2)(\beta+3)}\cdot\frac{3x^2}{6} + \cdots$$

Again Diff, w.r.t "x"

$$\frac{d^3}{dx^3}\left[{}_1F_1(\alpha,\beta;x)\right] = 0 + \frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)}$$

$$+ \frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}{\beta(\beta+1)(\beta+2)(\beta+3)}\frac{x}{2!} + \cdots$$

Similarly

$$\frac{d^p}{dx^p}\left[{}_1F_1(\alpha,\beta;x)\right] = \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+p-1)}{\beta(\beta+1)(\beta+2)\cdots(\beta+p-1)}\left[1 + \frac{(\alpha+p)}{(\beta+p)}x + \frac{(\alpha+p)(\alpha+p+1)}{(\beta+p)(\beta+p+1)}\frac{x^2}{2!} + \cdots\right]$$

$$= \frac{(\alpha)_p}{(\beta)_p}\left[\sum_{n=0}^{\infty}\frac{(\alpha+p)_n}{(\beta+p)_n}\frac{x^n}{n!}\right]$$

$$= \frac{(\alpha)_p}{(\beta)_p}\,{}_1F_1\left[\alpha+p,\ \beta+p,\ x\right]$$

Hence prove

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Question:- Show that

$$F(-n, \beta, \gamma, 1) = \frac{(\gamma-\beta)_n}{(\gamma)_n}$$

Solution:-

$$L.H.S = F(-n, \beta, \gamma, 1)$$

$$= \frac{\Gamma(\gamma)\,\Gamma(\gamma+n-\beta)}{\Gamma(\gamma+n)\,\Gamma(\gamma-\beta)}$$

$$= \frac{\Gamma[(\gamma-\beta)+n]}{\Gamma(\gamma-\beta)}$$

$$= \frac{\Gamma(\gamma-\beta)}{\Gamma(\gamma+n)}$$

$$\frac{\Gamma(\gamma)}{\Gamma(\gamma)}$$

$\Rightarrow$ using the property $\frac{\Gamma(a+n)}{\Gamma(a)} = (a)_n$

$$= \frac{(\gamma-\beta)_n}{(\gamma)_n} = R.H.S$$

$\Longleftrightarrow$

Question:- Show that

$$F(-n, 1-\beta-n, \alpha; 1) = \frac{1}{(\alpha)_n}\,\frac{(\alpha+\beta-1)_{2n}}{(\alpha+\beta-1)_n}$$

Solution:- Take L.H.S

$$L.H.S = F(-n, 1-\beta-n, \alpha; 1)$$

$$= \frac{\Gamma(\alpha)\,\Gamma(\alpha+n-1+\beta+n)}{\Gamma(\alpha+n)\,\Gamma(\alpha-1+\beta+n)}$$

$$= \left[\frac{\Gamma(\alpha)}{\Gamma(\alpha+n}\right] \frac{\Gamma(\alpha+\beta-1+2n)}{\Gamma(\alpha+\beta-1)} \times \frac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha+\beta-1+n)}$$

$$= \frac{1}{(\alpha)_n}\,\frac{(\alpha+\beta-1)_{2n}}{(\alpha+\beta-1)_n} = R.H.S$$

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Question:- Show that

$$F(-n, \alpha+n, \gamma, 1) = \frac{(-1)^n}{(\gamma)_n}\left[(1+\alpha-\gamma)_n\right]$$

Solution:- As

$$F(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

$$F(-n, \alpha+n, \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma+n-\alpha-n)}{\Gamma(\gamma+n)\Gamma(\gamma-\alpha-n)}$$

$$\cdot \frac{\Gamma(\overline{\gamma-\alpha-n} + n)}{\Gamma(\overline{\gamma-\alpha-n})}$$

$$= \frac{\Gamma(\gamma-\alpha-n)}{\Gamma(\gamma+n)}$$

$$\overline{\Gamma(\gamma)}$$

Now using $\frac{\Gamma(a+n)}{\Gamma(a)} = (a)_n$

$$= \frac{(\gamma-\alpha-n)_n}{(\gamma)_n}$$

$$= \frac{1}{(\gamma)_n}\left[(\gamma-\alpha-n)_n\right]$$

$$= \frac{1}{(\gamma)_n}\left[(\gamma-\alpha-n)(\gamma-\alpha-n+1)\cdots\right.$$
$$\left.(\gamma-\alpha-n+n-1)\right]$$

$$= \frac{1}{(\gamma)_n}\left[(\gamma-\alpha-n)(\gamma-\alpha-n+1)\cdots\right.$$
$$\left.(\gamma-\alpha-n+n-2)(\gamma-\alpha-n+n-1)\right]$$

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$$= \frac{1}{(\gamma)_n}\left[(\gamma-\alpha-n)(\gamma-\alpha-n+1)\cdots(\gamma-\alpha-2)(\gamma-\alpha-1)\right]$$

$$= \frac{1}{(\gamma)_n}\left[(\gamma-\alpha-1)(\gamma-\alpha-2)\cdots(\gamma-\alpha-n)\right]$$

$$= \frac{1}{(\gamma)_n}\left[(-)(1+\alpha-\gamma)(-1)(\overline{1+\alpha-\gamma}+1)\cdots(-1)\left[\overline{1+\alpha-\gamma}+(n-1)\right]\right]$$

$$= \frac{(-1)^n}{(\gamma)_n}\left[(1+\alpha-\gamma)(\overline{1+\alpha-\gamma}+1)\cdots\left[\overline{1+\alpha-\gamma}+(n-1)\right]\right]$$

$$= \frac{(-1)^n (1+\alpha-\gamma)_n}{(\gamma)_n}$$ Hence proove.

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Thanks

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LAPLACE TRANSFORMATION

Definition:- The Laplace transformation of a function $f(t)$ for $t > 0$ is defined by the following integral over the interval $[0, \infty)$

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t) \cdot dt$$

The resulting expression is a function of s, which we write as $F(s)$.

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) \cdot dt = \lim_{p \to \infty} \int_0^p e^{-st} f(t) \cdot dt$$

Theorem:- The Laplace transformation operator $\mathcal{L}$ is linear operator.

If "a" and "b" are constants while $g(t)$ and $h(t)$ are function of "t". Then

$$\mathcal{L}\{a\,g(t) + b\,h(t)\} = a\mathcal{L}\{g(t)\} + b\mathcal{L}\{h(t)\}$$

Proof:-

Let $f(t) = a\,g(t) + b\,h(t)$

Then

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}\{a\,g(t) + b\,h(t)\} \cdot dt$$

$$= a\int_0^\infty e^{-st} g(t)\,dt + b\int_0^\infty e^{-st} h(t)\,dt$$

$$= a\,\mathcal{L}\{g(t)\} + b\,\mathcal{L}\{h(t)\}$$

Example: $\mathcal{L}\{5t^2 + 7t^3\} = 5\mathcal{L}\{t^2\} + 7\mathcal{L}\{t^3\}$

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LAPLACE TRASFORM of Some Functions

1) - Laplace transform of a Constant.

Let $f(t) = K$ (some constant)

By Definition

$$\mathcal{L}\{K\} = \int_0^\infty e^{-st} K \cdot dt$$

$$= K \int_0^\infty e^{-st} \cdot dt$$

$$= K \left[\frac{e^{-st}}{-s} \Big|_0^\infty \right.$$

$$= \frac{-K}{s} \left[\frac{1}{e^{st}} \Big|_0^\infty \right.$$

$$= \frac{-K}{s} \left[\frac{1}{e^\infty} - \frac{1}{e^0} \right]$$

$$= \frac{-K}{s} [0 - 1]$$

$$= -\frac{K}{s}(-1) = \frac{K}{s}$$

$$\boxed{\mathcal{L}\{K\} = \frac{K}{s}}$$

Example : Laplace Trans form of "1"

Take $K = 1$ and put in above result

$$\mathcal{L}\{1\} = 1/s$$

Example (ii) :- Laplac Trans form of "6"

Take $K = 6$ and put in result (1)

$$\mathcal{L}\{6\} = 6/s$$

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Gamma Function :-

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} \cdot dx = (n-1)!$$

Gamma function is a very helpful for finding the results of Laplace Transformation.

(2) Laplace transform of "t".

Let $f(t) = t$

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st} \cdot t \cdot dt \quad \text{—— (i)}$$

put $st = x$

$t = x/s$

$dt = dx/s$

As $t \to 0$ Then $x \to 0$

$t \to \infty$ Then $x \to \infty$

Substituting these in (i)

$$\mathcal{L}\{t\} = \int_0^\infty e^{-x} \cdot \frac{x}{s} \cdot \frac{dx}{s}$$

$$= \frac{1}{s^2} \int_0^\infty e^{-x} \cdot x^{2-1} \cdot dx$$

By using Gamma Notation

$$= \frac{1}{s^2} \Gamma(2)$$

$$\boxed{\mathcal{L}\{t\} = \frac{1}{s^2}}$$

$\therefore \Gamma(2) = 1! = 1$

(3) Laplace Transform of "t^2".

Let $f(t) = t^2$

$$\mathcal{L}\{t^2\} = \int_0^\infty e^{-st} \cdot t^2 \cdot dt \quad \text{—— (i)}$$

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Put $st = x \Rightarrow t = \frac{x}{s}$

$dt = dx/s$

As $t \to 0, \infty$ Then $x \to 0, \infty$

Susituting these in (i)

$$\mathcal{L}\{t^2\} = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^2 \cdot \frac{dx}{s}$$

$$= \int_0^\infty e^{-x} \frac{x^2}{s^2} \cdot \frac{dx}{s}$$

$$= \frac{1}{s^3} \int_0^\infty e^{-x} x^{3-1} \cdot dx$$

By using Gamma function

$$= \frac{1}{s^3} \Gamma(3)$$

$$\mathcal{L}(t^2) = \frac{2!}{s^3} \qquad (\because \Gamma(3) = 2!)$$

(4) Laplace Transform of "t^n" $n \in \mathbb{Z}$

Let $f(t) = t^n$

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} t^n \cdot dt \quad \text{—(i)}$$

Put $st = x \Rightarrow t = x/s$

$dt = dx/s$

As $t \to 0, \infty$ Then $x \to 0, \infty$

subsituting these in (i)

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-x} \left[\frac{x}{s}\right]^n \cdot \frac{dx}{s}$$

$$= \int_0^\infty e^{-x} \frac{x^n}{s^n} \cdot \frac{dx}{s}$$

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$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} \, x^{(n+1)-1} \cdot dx$$

$$= \frac{1}{s^{n+1}} \Gamma(n+1)$$

$$\boxed{\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}} \qquad (\because \Gamma(n+1) = n!)$$

(5) Laplace Transform of e^{kt}

Let $f(t) = e^{kt}$

$$\mathcal{L}\{e^{kt}\} = \int_0^\infty e^{-st} \cdot e^{kt} \cdot dt$$

$$= \int_0^\infty e^{-(s-k)t} \cdot dt \quad \text{——(i)}$$

Put $(s-k)t = x \Rightarrow t = \frac{x}{s-k}$

$$dt = \frac{dx}{s-k}$$

And $t \to 0, \infty$ Then $x \to 0, \infty$

$$\mathcal{L}\{e^{kt}\} = \int_0^\infty e^{-x} \frac{dx}{s-k}$$

$$= \frac{1}{s-k} \int_0^\infty e^{-x} \, x^{1-1} \cdot dx$$

$$= \frac{1}{s-k} \Gamma(1)$$

$$\boxed{\mathcal{L}\{e^{kt}\} = \frac{1}{(s-k)}} \qquad (\because \Gamma(1) = 0! = 1)$$

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(6) Laplace transform of Sin Kt or Cos Kt

As we know $e^{iKt} = \cos Kt + i \sin Kt$

$$\mathcal{L}\{e^{iKt}\} = \int_0^\infty e^{-st} \cdot e^{iKt} \cdot dt$$

$$= \int_0^\infty e^{-(s-iK)t} \cdot dt \quad \text{---(i)}$$

Put, $(s-iK)t = x \Rightarrow t = x/s-iK$

$$dt = dx/s-iK$$

And x $t \to 0, \infty$ Then $x \to 0, \infty$

$$\mathcal{L}\{e^{iKt}\} = \int_0^\infty e^{-x} \frac{dx}{s-iK}$$

$$= \frac{1}{s-iK} \int_0^\infty e^{-x} x^{1-1} \cdot dx$$

$$= \frac{1}{s-iK} \Gamma(1)$$

$$= \frac{1}{s-iK} \times \frac{s+iK}{s+iK} \qquad \Gamma(1) = 0! = 1$$

$$= \frac{s+iK}{s^2-i^2K^2} = \frac{s+iK}{s^2+K^2} \qquad \therefore i^2 = -1$$

$$\mathcal{L}\{\cos Kt + i \sin Kt\} = \frac{s+iK}{s^2+K^2}$$

Since Laplace operator $\mathcal{L}$ is Linear

$$\mathcal{L}\{\cos Kt\} + i\mathcal{L}\{\sin Kt\} = \frac{s+iK}{s^2+K^2}$$

Comparing Real and Imaginary Part

$$\boxed{\mathcal{L}\{\cos Kt\} = \frac{s}{s^2+K^2}} \quad \& \quad \boxed{\mathcal{L}\{\sin Kt\} = \frac{K}{s^2+K^2}}$$

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(7) Laplace Transformation of Sin h Kt

As we know that

$$\text{Sin}\,hKt = \frac{e^{Kt} - e^{-Kt}}{2}$$

$$\mathcal{L}\{\text{Sin}\,hKt\} = \mathcal{L}\left\{\frac{e^{Kt} - e^{-Kt}}{2}\right\}$$

Since $\mathcal{L}$ is linear operator

$$= \frac{1}{2}\left[\mathcal{L}\{e^{Kt}\} - \mathcal{L}\{e^{-Kt}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{S-K} - \frac{1}{S+K}\right]$$

$$= \frac{1}{2}\left[\frac{S+K - S+K}{(S-K)(S+K)}\right]$$

$$= \frac{1}{2}\left[\frac{2K}{S^2-K^2}\right]$$

$$\boxed{\mathcal{L}\{\text{Sin}\,h\,Kt\} = \frac{K}{S^2-K^2}}$$

(8) Laplace Transformation of Cos h Kt

As we know $\text{Cos}\,hKt = \frac{e^{Kt} + e^{-Kt}}{2}$

$$\mathcal{L}\{\text{Cos}\,h\,Kt\} = \mathcal{L}\left\{\frac{e^{Kt} + e^{-Kt}}{2}\right\}$$

Since $\mathcal{L}$ is linear operator

$$= \frac{1}{2}\left[\mathcal{L}\{e^{Kt}\} + \mathcal{L}\{e^{-Kt}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{S-K} + \frac{1}{S+K}\right]$$

$$= \frac{1}{2}\left[\frac{S+K+S-K}{(S-K)(S+K)}\right]$$

$$= \frac{1}{2}\left[\frac{2S}{S^2-K^2}\right]$$

$$\boxed{\mathcal{L}\{\text{Cosh}\,Kt\} = \frac{S}{S^2-K^2}}$$

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A/2010

First Shifting Or First Translation Theorem

The first shifting theorem is a useful tool when faced with the challenge of Taking the Laplace transform of the product of exponential function with anothes function.

Let $f(t)$ be a piecewise continous function of exponential order "c" Then

$$\mathcal{L}\{e^{Kt} f(t)\} = F(s-K) = \mathcal{L}\{f(t)\}_{s \to s-K}$$

Proof:- $\mathcal{L}\{e^{Kt} f(t)\} = \int_0^\infty e^{-st} \cdot e^{Kt} f(t)\, dt$

$$= \int_0^\infty e^{-(s-K)t} \cdot f(t) \cdot dt$$

put $s' = s - K$

$$= \int_0^\infty e^{-s't} \cdot f(t) \cdot dt$$

$$= F(s')$$

$$= F(s-K)$$

OR $= \mathcal{L}\{f(t)\}_{s \to s-K}$

Example:-

(i) $\mathcal{L}\{e^{3t} f(t)\} = F(s-3)$

(ii) $\mathcal{L}\{e^{6t} f(t)\} = F(s-6)$

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Rule of Scales :- It enables us to calculate laplace transform of a function of the form $f(at)$ where $a>0$ is constant. It stats that

$$\mathcal{L}\{f(at)\} = \frac{1}{a} F\left[\frac{s}{a}\right]$$

Proof :-

$$\mathcal{L}\{f(at)\} = \int_0^\infty e^{-st} f(at)\,.dt \qquad a>0$$

Put $at = t'$

$$t = \frac{t'}{a} \Rightarrow dt = \frac{dt'}{a}$$

And $t \to 0$ then $t' \to 0$

$t \to \infty$ Then $t' \to \infty$

$$\mathcal{L}\{f(at)\} = \int_0^\infty e^{-(\frac{s}{a})t'} f(t').\frac{dt'}{a}$$

Let $\frac{s}{a} = s'$

$$= \frac{1}{a}\int_0^\infty e^{-s't'} f(t').dt'$$

$$= \frac{1}{a} F(s') = \frac{1}{a} F\{\tfrac{s}{a}\}$$

Laplace Transforms of Derivatives of a function

Theorem :- If $f(t)$ is continous and $f'(t)$ is piecewise continous on the interval $[0,\infty)$ and both are of exponential order i.e both of order e^{ax}

Then $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

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Proof :- As $f'(t)$ is a piecewise continuous on every finite closed interval and of exponential order a, so $\mathcal{L}\{f'(t)\}$ exist for $s > a$

By Definition

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\, dt$$

integration by parts

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty f(t)\, e^{-st}(-s)\, dt$$

$$= \left[\frac{f(t)}{e^{st}}\right]_0^\infty + s\int_0^\infty e^{-st} f(t)\cdot dt$$

$$= \left[\frac{f(t)}{e^\infty} - \frac{f(0)}{e^0}\right] + s\int_0^\infty e^{-st} f(t)\cdot dt$$

$$= \left[0 - f(0)\right] + s\,F(s)$$

$$= sF(s) - f(0)$$

(ii) $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$

Proof :- By Definition

$$\mathcal{L}\{f''(t)\} = \int_0^\infty e^{-st} f''(t)\cdot dt$$

integration by parts

$$\mathcal{L}\{f''(t)\} = \left[e^{-st} f'(t)\right]_0^\infty - \int_0^\infty e^{-st}(-s) f'(t)\cdot dt$$

$$= \left[\frac{f'(t)}{e^{st}}\right]_0^\infty + s\int_0^\infty e^{-st} f'(t)\cdot dt$$

using above result

$$= \left[\lim_{t\to\infty}\frac{f'(t)}{e^\infty} - \frac{f'(0)}{e^0}\right] + s\left[sF(s) - f(0)\right]$$

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$= [0 - f'(0)] + s^2 F(s) - s f(0)$

$= s^2 F(s) - s f(0) - f'(0)$ proved

Generalization

$$\mathcal{L}\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) + \cdots - f^{n-1}(0)$$

Proof :- $\mathcal{L}\{f^n(t)\} = \int_0^\infty e^{-st} f^n(t) \cdot dt$

$= \left[e^{-st} f^{n-1}(t)\right|_0^\infty - \int_0^\infty f^{n-1}(t) e^{-st}(-s) \cdot dt$

$= [0 - f^{n-1}(0)] + s \int_0^\infty e^{-st} f^{n-1}(t) \cdot dt$

$= s\,\mathcal{L}\{f^{n-1}(t)\} - f^{n-1}(0)$

$= s[s f^{n-2}(t) - f^{n-2}(0)] - f^{n-1}(0)$

$= s^2 [f^{n-2}(t)] - s f^{n-2}(0) - f^{n-1}(0)$

$= s^2 [s f^{n-3}(t) - f^{n-3}(0)] - s f^{n-2}(0) - f^{n-1}(0)$

$= s^3 f^{n-3}(t) - s^2 f^{n-3}(0) - s f^{n-2}(0) - f^{n-1}(0)$

Continuing this process we get

$$\mathcal{L}\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots - f^{n-1}(0).$$

$$= s^n F(s) + \sum_{K=1}^{n} s^{n-K} f^{K-1}(0)$$

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Example:- calculate the Laplace Transformation of t^{α} where $\alpha \in R$

Solution:- Let $f(t) = t^{\alpha}$

By Definition

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) \cdot dt$$

$$\mathcal{L}\{t^{\alpha}\} = \int_0^{\infty} e^{-st} t^{\alpha} \cdot dt$$

put $st = x \Rightarrow t = \frac{x}{s}$

$$dt = \frac{dx}{s}$$

As $x \to 0, \infty$ when $t \to 0, \infty$

there for

$$\mathcal{L}\{t^{\alpha}\} = \int_0^{\infty} e^{-x} \left(\frac{x}{s}\right)^{\alpha} \frac{dx}{s}$$

$$= \int_0^{\infty} e^{-x} \frac{x^{\alpha}}{s^{\alpha}} \cdot \frac{dx}{s}$$

$$= \frac{1}{s^{\alpha+1}} \int_0^{\infty} e^{-x} x^{(\alpha+1)-1} \cdot dx$$

$$\boxed{\mathcal{L}\{t^{\alpha}\} = \frac{1}{s^{\alpha+1}} \Gamma(\alpha+1)} \qquad \alpha \in \mathbb{R}$$

Desire the Laplace transformation of $t^{1/2}$ and $t^{-1/2}$.

As we know that

$$\mathcal{L}\{t^{\alpha}\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}} \quad \text{---(1)}$$

put $\alpha = 1/2$ in (1)

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$$\mathcal{L}\{t^{1/2}\} = \frac{\Gamma(1/2+1)}{s^{\frac{1}{2}+1}}$$

$$= \frac{\frac{1}{2}\Gamma(1/2)}{s^{3/2}}$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} \qquad \left(\because \Gamma(1/2) = \sqrt{\pi}\right)$$

Now put $\alpha = -1/2$ in (1)

$$\mathcal{L}\{t^{-1/2}\} = \frac{\Gamma(\frac{-1}{2}+1)}{s^{\frac{-1}{2}+1}}$$

$$= \frac{\Gamma(1/2)}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}$$

Laplace Transformation of "$t^n \cdot e^{kt}$"

Solution:-

Let $f(t) = t^n e^{kt}$.

By Definition

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \cdot [t^n e^{kt}] \cdot dt$$

$$= \int_0^\infty e^{-(s-k)t} \cdot t^n \cdot dt \qquad (1)$$

put $(s-k)t = x$

$$t = x/(s-k)$$

$$dt = dx/(s-k)$$

And $x \to 0$ when $t \to 0$

$x \to \infty$ when $t \to \infty$

Subsituting these in (1)

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$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-x}\left[\frac{x}{s-K}\right]^n \cdot \frac{dx}{[s-K]}$$

$$= \frac{1}{(s-K)^{n+1}} \int_0^\infty e^{-x} x^n \cdot dx$$

$$= \frac{1}{(s-K)^{n+1}} \Gamma(n+1)$$

$$\boxed{\mathcal{L}\{t^n e^{Kt}\} = \frac{n!}{(s-K)^{n+1}}, \quad s > K}$$

Example:- Evaluate the L-T

(i) te^{Kt} (ii) t^2e^{3t}

Solution:-

As we know that

$$\mathcal{L}\{t^n e^{Kt}\} = \frac{n!}{(s-K)^{n+1}} \quad \text{---} \quad (A)$$

put $n = 1$

$$\mathcal{L}\{te^{Kt}\} = \frac{1!}{(s-K)^{1+1}}$$

$$= \frac{1}{(s-K)^2}$$

Solution (ii) $\mathcal{L}\{t^2e^{3t}\}$

Put $n = 2$ & $K = 3$ in (A)

$$= \frac{2!}{(s-3)^3}$$

$$\mathcal{L}\{t^2e^{3t}\} = \frac{2}{(s-3)^3}$$

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Laplace Transformation of (i) $e^{at}\sin bt$ and (ii) $e^{at}\cos bt$

As we know that $e^{ibt} = \cos bt + i\sin bt$

And $e^{at}\left[e^{ibt}\right] = e^{at}\left[\cos bt + i\sin bt\right]$

$e^{(a+ib)t} = e^{at}\cos bt + ie^{at}\sin bt$

$e^{at}\cos bt + ie^{at}\sin bt = e^{(a+ib)t}$

Now Apply Laplace Transformation on both sid

$\mathcal{L}\{e^{at}\cos bt + ie^{at}\sin bt\} = \mathcal{L}\left[e^{(a+ib)t}\right]$

Since $\mathcal{L}\{e^{kt}\} = 1/s-k$

$\mathcal{L}\{e^{at}\cos bt + ie^{at}\sin bt\} = \dfrac{1}{s-(a+ib)}$

Since $\mathcal{L}$ is Linear operator

$= \dfrac{1}{(s-a)+ib} \times \dfrac{(s-a)-ib}{(s-a)-ib}$

$= \dfrac{(s-a)-ib}{(s-a)^2 - i^2b^2}$

$\mathcal{L}\{e^{at}\cos bt\} + i\mathcal{L}\{e^{at}\sin bt\} = \dfrac{(s-a)-bi}{(s-a)^2+b^2}$

Compering the Real and imaginary part

$\mathcal{L}\{e^{at}\cos bt\} = \dfrac{(s-a)}{(s-a)^2+b^2}$; $s > a$

$\mathcal{L}\{e^{at}\sin bt\} = \dfrac{b}{(s-a)^2+b^2}$; $s > a$

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A/2011

Question :- $\mathcal{L}\{t^{\frac{3}{2}} e^{t}\}$

As we know that

$$\mathcal{L}\{t^n e^{Kt}\} = \frac{n!}{(s-K)^{n+1}} = \frac{\Gamma(n+1)}{(s-K)^{n+1}}$$

put $n = \frac{3}{2}$ and $K = 1$

$$\mathcal{L}\{t^{3/2} e^{t}\} = \frac{\Gamma(\frac{3}{2}+1)}{(s-1)^{\frac{3}{2}+1}}$$

$$= \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma(1/2)}{(s-1)^{5/2}}$$

$$= \frac{3\sqrt{\pi}}{4(s-1)^{5/2}} \quad (\because \Gamma(1/2) = \sqrt{\pi})$$

5/2010

Question :- $\mathcal{L}\{e^{3/2} e^{4t}\}$

As we know that

$$\mathcal{L}\{t^n e^{Kt}\} = \frac{\Gamma(n+1)}{(s-K)^{n+1}}$$

put $n = 3/2$ and $K = 4$

$$\mathcal{L}\{e^{3/2} e^{4t}\} = \frac{\Gamma(3/2+1)}{(s-4)^{\frac{3}{2}+1}}$$

$$= \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma(1/2)}{(s-4)^{5/2}}$$

$$= \frac{3\sqrt{\pi}}{4(s-4)^{5/2}}$$

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A/2010

Question :- $\mathcal{L}\{\sqrt{t}\, e^{-5t}\}$

As we know

$$\mathcal{L}\{t^n e^{kt}\} = \frac{\Gamma(n+1)}{(s-k)^{n+1}} \quad \text{(A)}$$

put $n = 1/2$ and $k = -5$

$$\mathcal{L}\{\sqrt{t}\, e^{-5t}\} = \frac{\Gamma(1/2+1)}{[s-(-5)]^{\frac{1}{2}+1}}$$

$$= \frac{\frac{1}{2}\Gamma(1/2)}{(s+5)^{3/2}}$$

$$= \frac{\sqrt{\pi}}{2\cdot[s+5]^{3/2}} \qquad \left[\because \Gamma 1/2 = \sqrt{\pi}\right]$$

A/2009

Question :- $\mathcal{L}\{\sqrt{t}\, e^{-2t}\}$

put $n = 1/2$ and $k = -2$ in (A)

$$\mathcal{L}\{\sqrt{t}\, e^{-2t}\} = \frac{\Gamma(\frac{1}{2}+1)}{[s-(-2)]^{\frac{1}{2}+1}}$$

$$= \frac{\frac{1}{2}\Gamma(1/2)}{(s+2)^{3/2}} = \frac{\sqrt{\pi}}{2(s+2)}$$

S/2009

Question :- $\mathcal{L}\{\sqrt{t}\, e^{-3t}\}$

put $n = 1/2$ and $k = -3$ in (A)

$$\mathcal{L}\{\sqrt{t}\, e^{-2t}\} = \frac{\Gamma(\frac{1}{2}+1)}{[s-(-3)]^{\frac{1}{2}+1}}$$

$$= \frac{\frac{1}{2}\Gamma(1/2)}{(s+3)^{3/2}}$$

$$= \frac{\sqrt{\pi}}{2(s+3)^{3/2}}$$

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Q:- $\mathcal{L}\{t \sin Kt\}$ and $\mathcal{L}\{t \cos Kt\}$

Solution:- As $\cos Kt + i \sin Kt = e^{iKt}$

$$t[\cos Kt + i \sin Kt] = te^{iKt}$$

Apply Laplace Transformation

$$\mathcal{L}\{t\cos Kt + it \sin Kt\} = \mathcal{L}\{te^{iKt}\}$$

As we know

$$\mathcal{L}\{t^n e^{Kt}\} = \frac{n!}{(s-K)^{n+1}}$$

$$\mathcal{L}\{t\cos Kt + it \sin Kt\} = \frac{1}{(s-ik)^2}$$

$$= \frac{1}{s^2 + i^2k^2 - 2ski}$$

$$= \frac{1}{(s^2-k^2) - i2sk} \times \frac{(s^2-k^2) + i2sk}{(s^2-K^2) + i2sK}$$

$$\mathcal{L}\{t\cos Kt\} + i\mathcal{L}\{t \sin Kt\} = \frac{(s^2-K^2) + i2sK}{(s^2-K^2)^2 + 4s^2K^2}$$

Comparing Real and imaginary part

$$\mathcal{L}\{t\cos Kt\} = \frac{s^2-K^2}{(s^2-K^2)^2 + 4s^2K^2}$$

$$\mathcal{L}\{t \sin Kt\} = \frac{2sK}{(s^2-K^2)^2 + 4s^2K^2}$$

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5/2011

Question:- Find (i) $\mathcal{L}\{t\sin t\}$

(ii) $\mathcal{L}\{t\cos t\}$

Solution:- As $\cos t + i\sin t = e^{it}$

$$t[\cos t + i\sin t] = te^{it}$$

Apply Laplace Transformation

$$\mathcal{L}[t\cos t + it\sin t] = \mathcal{L}[te^{it}]$$

As we know :-

$$\mathcal{L}\{t^n e^{kt}\} = \frac{n!}{(s-k)^{n+1}}$$

Put $n=1$ and $k=i$

$$\mathcal{L}\{t\cos t\} + i\,\mathcal{L}\{t\sin t\} = \frac{1}{(s-i)^2}$$

$$= \frac{1}{s^2+i^2-2is}$$

$$= \frac{1}{(s^2-1)-2is} \times \frac{(s^2-1)+2is}{(s^2-1)+2is}$$

$$= \frac{(s^2-1)+2is}{(s^2-1)^2-4i^2s^2}$$

$$= \frac{(s^2-1)+2is}{s^4+1-2s^2+4s^2}$$

$$= \frac{(s^2-1)+2is}{s^4+1+2s^2}$$

$$\mathcal{L}\{t\cos t\} + i\,\mathcal{L}\{t\sin t\} = \frac{(s^2-1)+2is}{(s^2+1)^2}$$

Comparing Real and imaging part

$$\mathcal{L}\{t\cos t\} = \frac{(s^2-1)}{(s^2+1)^2}$$

$$\mathcal{L}\{t\sin t\} = \frac{2s}{(s^2+1)^2}$$

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Another Method :- $\mathcal{L}\{t\sin t\}$

As we know that

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f(t)\}$$

Here $n=1$ and $f(t)=\sin t$

$$\mathcal{L}\{t\sin t\} = (-1)^1 \frac{d}{ds}\mathcal{L}\{\sin t\}$$

$$= (-1)\frac{d}{ds}\left[\frac{1}{s^2+1}\right]$$

$$= (-1)\frac{d}{ds}(s^2+1)^{-1}$$

$$= (-1)(-1)(s^2+1)^{-2}(2s)$$

$$\mathcal{L}\{t\sin t\} = \frac{2s}{(s^2+1)^2}$$

5/2011

Question: $\mathcal{L}\{t\sin 2t\}$ and $\mathcal{L}\{t\cos 2t\}$

Solution:- As we know

$$\cos 2t + i\sin 2t = e^{2it}$$

$$t[\cos 2t + i\sin 2t] = te^{2it}$$

$$t\cos 2t + it\sin 2t = [te^{2it}]$$

Apply Laplace Transformation

$$\mathcal{L}\{t\cos 2t\} + i\mathcal{L}\{t\sin 2t\} = \mathcal{L}\{te^{2it}\}$$

As we know

$$\mathcal{L}\{t^n e^{kt}\} = \frac{n!}{(s-k)^{n+1}}$$

put $n=1$ & $k=2i$

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$$= \left[\frac{1}{(s-2i)^2}\right]$$

$$= \left[\frac{1}{s^2+4i^2-4is}\right]$$

$$= \left[\frac{1}{(s^2-4)-4is}\right]$$

$$= \left[\frac{1}{(s^2-4)-4is} \times \frac{(s^2-4)+4is}{(s^2-4)+4is}\right]$$

$$= \left[\frac{(s^2-4)+4is}{(s^2-4)^2-16i^2s^2}\right]$$

$$= \left[\frac{(s^2-4)+4is}{s^4+16-8s^2+16s^2}\right]$$

$$= \left[\frac{(s^2-4)+4is}{s^4+16+8s^2}\right]$$

$$= \left[\frac{(s^2-4)+4is}{(s^2+4)^2}\right]$$

$$\mathcal{L}\{t\cos 2t\}+i\mathcal{L}\{t\sin 2t\} = \left[\frac{(s^2-4)}{(s^2+4)^2} + i\frac{4s}{(s^2+4)^2}\right]$$

Comparing imaginary part

$$\mathcal{L}\{t\sin 2t\} = \frac{4s}{(s^2+4)^2}$$

If we compare Real Part then

$$\mathcal{L}\{t\cos(2t)\} = \frac{s^2-4}{(s^2+4)^2}$$

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Exercise :- Compute Laplace Transformation

(1) - $\sin^2 Kt$

Let $f(t) = \sin^2 Kt$

Since $\sin^2 Kt = \frac{1 - \cos 2Kt}{2}$

$$\mathcal{L}\{\sin^2 Kt\} = \mathcal{L}\left\{\frac{1-\cos 2Kt}{2}\right\}$$

$$= \frac{1}{2}\left[\mathcal{L}\{1 - \cos 2Kt\}\right]$$

$$= \frac{1}{2}\left[\mathcal{L}\{1\} - \mathcal{L}\{\cos 2Kt\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4K^2}\right]$$

$$= \frac{1}{2}\left[\frac{s^2+4K^2 - s^2}{s(s^2+4K^2)}\right]$$

$$= \frac{1}{2}\left[\frac{4K^2}{s(s^2+4K^2)}\right]$$

$$\mathcal{L}\{\sin^2 Kt\} = \frac{2K^2}{s(s^2+4K^2)}$$

---

(2) - $\mathcal{L}\{\sin^2 t\}$

As we know that by above Question

$$\mathcal{L}\{\sin^2 Kt\} = \frac{2K^2}{s(s^2+4K^2)} \quad \text{---(i)}$$

Put $K = 1$ in (i)

$$\mathcal{L}\{\sin^2 t\} = \frac{2}{s(s^2+4)}$$

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(3) - $\mathcal{L}\{\cos^2 Kt\}$

As we know that $\cos^2 Kt = \frac{1+\cos 2Kt}{2}$

$$\mathcal{L}\{\cos^2 Kt\} = \mathcal{L}\left\{\frac{1+\cos 2Kt}{2}\right\}$$

$$= \frac{1}{2}\left[\mathcal{L}\{1+\cos 2Kt\}\right]$$

$$= \frac{1}{2}\left[\mathcal{L}\{1\} + \mathcal{L}\{\cos 2Kt\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4K^2}\right]$$

$$= \frac{1}{2}\left[\frac{s^2+4K^2+s^2}{s(s^2+4K^2)}\right]$$

$$= \frac{1}{2}\left[\frac{2s^2+4K^2}{s(s^2+4K^2)}\right]$$

$$= \frac{1}{2}\left[\frac{2(s^2+2K^2)}{s(s^2+4K^2)}\right]$$

$$\mathcal{L}\{\cos^2 Kt\} = \frac{s^2+2K^2}{s(s^2+4K^2)}$$

(4) - $\mathcal{L}\{\cos^2 t\}$

As by above question

$$\mathcal{L}\{\cos^2 Kt\} = \frac{s^2+2K^2}{s(s^2+4K^2)} \quad \text{---- (i)}$$

Put $K = 1$ in (i)

$$\mathcal{L}\{\cos^2 t\} = \frac{s^2+2}{s(s^2+4)}$$

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(5) - $\mathcal{L}\{4 + e^{3t} - 5e^{-8t}\}$

Since $\mathcal{L}$ is linear operator

$$\mathcal{L}\{4 + e^{3t} - 5e^{-8t}\} = 4\mathcal{L}\{1\} + \mathcal{L}\{e^{3t}\} - 5\mathcal{L}\{e^{-8t}\}$$

$$= 4\left[\frac{1}{s}\right] + \left[\frac{1}{(s-3)}\right] - 5\left[\frac{1}{s+8}\right]$$

$$= \frac{4}{s} + \frac{1}{s-3} - \frac{5}{s+8}$$

(6) - $\mathcal{L}[t + e^t]^3$

$$(t + e^t)^3 = t^3 + e^{3t} + 3t^2e^t + 3te^{2t}$$

$$\mathcal{L}\{(t+e^t)^3\} = \mathcal{L}\{t^3 + e^{3t} + 3t^2e^t + 3te^{2t}\}$$

$$= \mathcal{L}\{t^3\} + \mathcal{L}\{e^{3t}\} + 3\mathcal{L}\{t^2e^t\} + 3\mathcal{L}\{te^{2t}\}$$

$$= \frac{3!}{s^4} + \frac{1}{s-3} + 3\cdot\frac{2}{(s-1)^2} + 3\cdot\frac{1}{(s-2)^2}$$

$$= \frac{6}{s^4} + \frac{1}{s-3} + \frac{6}{(s-1)^2} + \frac{3}{(s-2)^2}$$

(7) - $\mathcal{L}\{4e^{-7t/2}\}$

$$\Rightarrow 4\mathcal{L}\{e^{\frac{-7t}{2}}\}$$

As we know $\mathcal{L}\{e^{kt}\} = \frac{1}{s-k}$

$$\Rightarrow 4\left[\frac{1}{s+\frac{7}{2}}\right] = 4\left[\frac{1}{\frac{2s+7}{2}}\right]$$

$$= 4\left[\frac{2}{2s+7}\right] = \frac{8}{2s+7}$$

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(8)- $\mathcal{L}\{e^{-2t}\sin 3t\}$

As By First shifting Theorem

$$\mathcal{L}\{e^{kt}f(t)\} = F\{s-k\} = \mathcal{L}\{f(t)\}_{s\to s-1}$$

So

$$\mathcal{L}\{e^{-2t}\sin 3t\} = \mathcal{L}\{\sin 3t\}_{s\to s+2}$$

$$= \frac{3}{(s+2)^2+9}$$

(9)- $\mathcal{L}\{e^{-3t}\cos 2t\}$

As by First shifting Theorem

$$\mathcal{L}\{e^{kt}f(t)\} = F\{s-k\} = \mathcal{L}\{f(t)\}_{s\to s-k}$$

$$\mathcal{L}\{e^{-3t}\cos 2t\} = \mathcal{L}\{\cos 2t\}_{s\to s+3}$$

$$= \frac{(s+3)}{(s+3)^2+4}$$

(10)- $\mathcal{L}\{e^{2t}\cosh 4t\}$

By First shifting theorem

$$\mathcal{L}\{e^{kt}f(t)\} = \mathcal{L}\{f(t)\}_{s\to s-k}$$

$$\mathcal{L}\{e^{2t}\cosh 4t\} = \mathcal{L}\{\cosh 4t\}_{s\to s-2}$$

$$= \frac{(s-2)}{(s-2)^2-16}$$

$$= \frac{s-2}{s^2-4s+4-16} = \frac{s-2}{s^2-4s-12}$$

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LAPLACE TRANSFORMATION OF LOGARITHMIC FUNCTION

Let $f(t) = \ln(t)$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t).\,dt$$

$$\mathcal{L}\{\ln t\} = \int_0^\infty e^{-st} . \ln t . dt$$

Put $st = x$

$$t = \frac{x}{s}$$

$$dt = dx/s$$

when $t \to 0, \infty$ Then $x \to 0, \infty$

$$\mathcal{L}\{\ln t\} = \int_0^\infty e^{-x} \ln\left(\frac{x}{s}\right) \cdot \frac{dx}{s}$$

$$= \frac{1}{s}\int_0^\infty e^{-x}[\ln x - \ln s]\,.\,dx$$

$$= \frac{1}{s}\int_0^\infty e^{-x}\ln x - \frac{1}{s}\int_0^\infty e^{-x}.\ln s\,.\,dx$$

$$= \frac{1}{s}\int_0^\infty e^{-x}.\ln x - \frac{\ln s}{s}\int_0^\infty e^{-x}.\,dx$$

$$= \frac{1}{s}\int_0^\infty e^{-x}.\ln x - \frac{\ln x}{x}\,\Gamma(2)$$

$$= \frac{1}{s}\int_0^\infty e^{-x}.\ln x - \frac{\ln x}{x} \quad (1) \quad \boxed{(\because \Gamma(2) = 1)}$$

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As we know that $\Gamma(u+1) = \int_0^\infty e^{-x} x^u \, dx$

Differentiating

$$\Gamma'(u+1) = \int_0^\infty e^{-x} \cdot x^u \cdot \ln x \cdot dx$$

Put $u=0$.

$$\Gamma'(1) = \int_0^\infty e^{-x} \ln x \cdot dx$$

putting this in (1)' :

$$\mathcal{L}\{\ln(t)\} = \frac{1}{s}\Gamma'(1) - \frac{\ln s}{s}$$

$$= \frac{\Gamma'(1) - \ln s}{s}$$

where $\Gamma'(1)$ is a constant called Euler's constant whose value is approximately 0.5772156649 ...

LAPLACE TRANSFORM of function "t" ftt

Theorem :- If $f(t)$ is a function of exponential order c.

Then $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$

Proof :- As we know that

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

Differentiating w.r.t "s"

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$$\frac{d}{dS}[F(S)] = \int_0^\infty \frac{d}{dS} e^{-St} f(t) \cdot dt$$

$$= \int_0^\infty e^{-St}(-t) f(t) \cdot dt$$

$$= (-1)\int_0^\infty e^{-St}\{t f(t)\} \cdot dt \quad \text{---(1)}$$

$$\frac{d}{dS}[F(S)] = (-1)\,\mathcal{L}\{t f(t)\}$$

Again Equation (1) Differentiate w.r.t 'S'

$$\frac{d^2}{dS^2}[F(S)] = (-1)\int_0^\infty e^{-St}(-t)\{t f(t)\}\,dt$$

$$\frac{d^2}{dS^2}[F(S)] = (-1)^2 \int_0^\infty e^{-St}\{t^2 f(t)\} \cdot dt$$

$$\frac{d^2}{dS^2}[F(S)] = (-1)^2\,\mathcal{L}\{t^2 f(t)\}$$

Continuing this process

$$\frac{d^n}{dS^n}[F(S)] = (-1)^n\,\mathcal{L}\{t^n f(t)\}$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{dS^n}[F(S)]$$

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TABLE OF LAPLACE TRANSFORM

	$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
1	1	$\frac{1}{s}$, $s>0$
2	t	$\frac{1}{s^2}$, $s>0$
3	t^n , $n \in \mathbb{Z}^+$	$\frac{n!}{s^{n+1}}$, $s>0$
4	t^α , $\alpha > -1$	$\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$, $s>0$
5	e^{Kt}	$\frac{1}{s-K}$, $s>K$
6	$\sin Kt$	$\frac{K}{s^2+K^2}$, $s>0$
7	$\cos Kt$	$\frac{s}{s^2+K^2}$, $s>0$
8	$\sinh Kt$	$\frac{K}{s^2-K^2}$, $s>\lvert K\rvert$
9	$\cosh Kt$	$\frac{s}{s^2-K^2}$, $s>\lvert K\rvert$
10	$e^{Kt}\sin at$	$\frac{a}{(s-K)^2+a^2}$, $s>K$
11	$e^{Kt}\cos at$	$\frac{s-K}{(s-K)^2+a^2}$, $s>K$
12	$e^{Kt}f(t)$	$F(s-K)$, $s>K$
13	$f^n(t)$	$s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{n-1}(0)$
14	$t^n e^{Kt}$	$\frac{n!}{(s-K)^{n+1}}$, $s>K$
15	$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n} F(s)$

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INVERSE LAPLACE TRANSFORM

If $F(s)$ is the Laplace transform of a function $f(t)$, then $f(t)$ is called the inverse Laplace transform of $F(s)$ and is denoted by

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

The inverse Laplace transform will be useful for solving initial value problems.

Theorem :- (Linearity property)

If $\mathcal{L}^{-1}\{F_1(s)\} = f_1(t)$

and $\mathcal{L}^{-1}\{F_2(s)\} = f_2(t)$ Then for any constant a, b

$$\mathcal{L}^{-1}\{a F_1(s) + b F_2(s)\} = a\mathcal{L}^{-1}\{F_1(s)\} + b\mathcal{L}^{-1}\{F_2(s)\}$$

$$= a f_1(t) + b f_2(t)$$

Examples :- Find the inverse Laplace transform of (i) $\frac{1}{s-8}$ (ii) $\frac{1}{3-s}$

Solution :- $F(s) = \frac{1}{s-8}$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-8}\right\}$$

$$f(t) = e^{8t}$$

(ii) $F(s) = \frac{1}{3-s} = \frac{-1}{s-3}$

$$\mathcal{L}^{-1}\{F(s)\} = -\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} = -e^{3t}$$

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Compute :- $\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s-15}\right\}$

Solution :- Here

$$\frac{1}{s^2+2s-15} = \frac{1}{s^2+5s-3s-15} = \frac{1}{s(s+5)-3(s+5)}$$

$$\frac{1}{s^2+2s-15} = \frac{1}{(s+5)(s-3)} = \frac{A}{s+5} + \frac{B}{s-3} \quad \text{---(1)}$$

Equation (1) multiply by $(s+5)(s-3)$ on both side

$$1 = A(s-3) + B(s+5) \quad \text{---(2)}$$

put $s = 3$ in (2)

$$1 = A(3-3) + B(3+5)$$

$$1 = 0 + B(8) \Rightarrow \boxed{B = 1/8}$$

put $s = -5$ in (2)

$$1 = A(-5-3) + B(-5+5)$$

$$1 = A(-8) + 0$$

$$\boxed{A = -1/8}$$

Putting the value of A and B in (1)

$$\frac{1}{s^2+2s-15} = \frac{-1}{8(s+5)} + \frac{1}{8(s-3)}$$

$$= \frac{1}{8}\left[\frac{1}{s-3} - \frac{1}{s+5}\right]$$

Therefore $\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s-15}\right\} = \mathcal{L}^{-1}\left[\frac{1}{8}\left\{\frac{1}{s-3} - \frac{1}{s+5}\right\}\right.$

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$$= \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} - \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}$$

$$= \frac{1}{8}e^{3t} - \frac{1}{8}e^{-5t} \quad \text{Answer}$$

4/2007

Example :- calculate $\mathcal{L}^{-1}\left\{\frac{s+4}{s^2+3s+2}\right\}$

Solution :- $\frac{s+4}{s^2+3s+2} = \frac{s+4}{s^2+2s+s+2}$

$$= \frac{s+4}{s(s+2)+1(s+2)}$$

$$= \frac{s+4}{(s+2)(s+1)}$$

$$\frac{s+4}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} \quad \text{---(1)}$$

multiply both side by $(s+1)(s+2)$, we get

$$s+4 = A(s+2) + B(s+1) \quad \text{---(2)}$$

put $s = -2$ in (2)

$$-2+4 = A(-2+2) + B(-2+1)$$

$$2 = 0 - B$$

$$\boxed{B = -2}$$

Now put $s = -1$ in (2)

$$-1+4 = A(-1+2) + B(-1+1)$$

$$\boxed{3 = A}$$

Putting these in (1)

$$\frac{s+4}{(s+1)(s+2)} = \frac{3}{s+1} - \frac{2}{s+2}$$

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$$\mathcal{L}^{-1}\left\{\frac{s+4}{(s+1)(s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{s+1} - \frac{2}{s+2}\right\}$$

$$= 3\mathcal{L}^{-1}\frac{1}{s+1} - 2\mathcal{L}^{-1}\frac{1}{s+2}$$

$$= 3e^{-t} - 2e^{-2t}$$

Example :- Compute $\mathcal{L}^{-1}\left\{\frac{2s}{s^2+2}\right\}$

$$2\mathcal{L}^{-1}\left\{\frac{s}{s^2+(\sqrt{2})^2}\right\}$$

$$2\cos\sqrt{2}t$$

Example :- $\mathcal{L}^{-1}\left\{\frac{s}{s^2+4s}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{s}{s(s+4)}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} = e^{-4t}$$

Example :- Evaluate $\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s}\right\}$

Here $\frac{1}{s^2+2s} = \frac{1}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2}$ — (i)

multiply equation by $s(s+2)$ on both side

$$1 = A(s+2) + B(s) \quad \text{— (ii)}$$

put $s=0$ in (ii)

$$1 = A(0+2) + B(0)$$

$$\boxed{A = 1/2}$$

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put $s+2=0 \Rightarrow s=-2$ in (ii)

$$1 = A(-2+2) + B(-2)$$

$$\boxed{B = -1/2}$$

Putting the values of A and B in (i)

$$\frac{1}{s(s+2)} = \frac{1}{2s} - \frac{1}{2(s+2)}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s(s+2)}\right] = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+2}\right]$$

$$= \frac{1}{2}\left[\mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+2}\right)\right]$$

$$= \frac{1}{2}\left[1 - e^{-2t}\right]$$

$$= \frac{1}{2}\left[1 - e^{-t} \cdot e^{-t}\right]$$

$$= \frac{1}{2}\left[1 - \frac{e^{-t}}{e^{t}}\right]$$

$$= \frac{1}{2}\left[\frac{e^{t} - e^{-t}}{e^{t}}\right]$$

$$= \frac{1}{e^{t}}\left[\frac{e^{t} - e^{-t}}{2}\right]$$

$$= e^{-t} \sinh t \quad \text{Answer}$$

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Example:- Find the inverse Laplace Transformation $F(S) = \frac{2S+3}{S^4}$

Solution :- $F(S) = \frac{2S+3}{S^4} = \frac{A}{S} + \frac{B}{S^2} + \frac{C}{S^3} + \frac{D}{S^4}$

Multipling S^4 on both side

$$2S + 3 = AS^3 + BS^2 + CS + D$$

put $S = 0$

$$\boxed{3 = D}$$

Equating the Co-efficient of S^3, S^2, S

$$A = B = 0$$

$$\boxed{C = 2}$$

$$F(S) = \frac{2}{S^3} + \frac{3}{S^4}$$

$$\mathcal{L}^{-1}\{F(S)\} = \mathcal{L}^{-1}\left\{\frac{2}{S^3} + \frac{3}{S^4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2!}{S^3} + \frac{1}{2}\frac{3!}{S^4}\right\}$$

$$= \mathcal{L}^{-1}\frac{2!}{S^3} + \frac{1}{2}\mathcal{L}^{-1}\frac{3!}{S^4}$$

$$= t^2 + \frac{t^3}{2} \quad \text{Answer}$$

(ii) $F(S) = \frac{1}{S} + \frac{2}{S^3} - \frac{7}{S^{5/2}}$

$$\mathcal{L}^{-1}\left[\frac{1}{S} + \frac{2}{S^3} - \frac{7}{S^{5/2}}\right] = \mathcal{L}^{-1}\left\{\frac{1}{S}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{S^3}\right\} - 7\mathcal{L}^{-1}\left\{\frac{1}{S^{5/2}}\right\}$$

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$$= 1 + t^2 - 7\,\mathcal{L}^{-1}\left\{\frac{\Gamma(3/2+1)}{s^{\frac{3}{2}+1}}\right\}$$

$$\Gamma(3/2+1)$$

$$= 1 + t^2 - \frac{7}{\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)}\,\mathcal{L}^{-1}\left\{\frac{\Gamma(1+3/2)}{s^{1+3/2}}\right\}$$

$$= 1 + t^2 - \frac{7}{\frac{3}{4}\sqrt{\pi}}\,t^{3/2}$$

$$= 1 + t^2 - \frac{28}{3\sqrt{\pi}}\,t^{3/2}$$

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* * ═══════ * *

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2-k^2)}\right\}$$

$$\frac{1}{s^2(s^2-k^2)} = \frac{1}{k^2}\left[\frac{1}{s^2-k^2} - \frac{1}{s^2}\right]$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2-k^2)}\right\} = \frac{1}{k^2}\left[\mathcal{L}^{-1}\frac{1}{(s^2-k^2)}\right] - \frac{1}{k^2}\left[\mathcal{L}^{-1}\frac{1}{s^2}\right]$$

$$= \frac{1}{k^2}\cdot\frac{1}{k}\,\mathcal{L}^{-1}\frac{k}{(s^2-k^2)} - \frac{1}{k^2}\,t$$

$$= \frac{1}{k^3}\sinh kt - \frac{t}{k^2}$$

* * ═══════ * *

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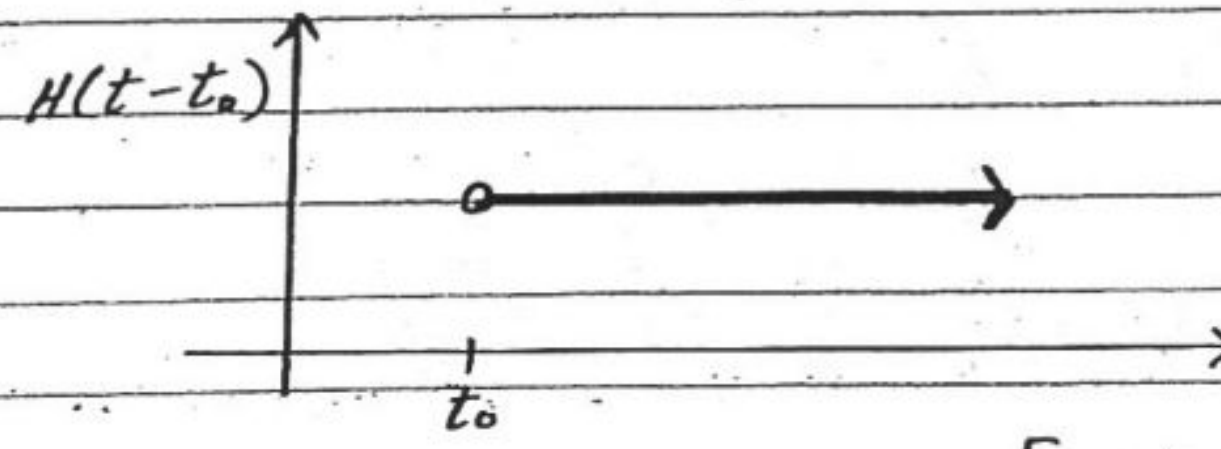
Unit Step Function

Unit step function is also known as Heaviside step function. And it is denoted by a symbol $H(t-t_0)$.

$$H(t-t_0) = \begin{cases} 1 & ; t > t_0 \\ 0 & ; t \leq t_0 \end{cases}$$

∴ It is clear that the step function is Discontinous at $t = t_0$

we represent the unit step length by graph as follows.



Similarly $t^3 H(t-t_0) = \begin{cases} t^3 & ; t > t_0 \\ 0 & ; t \leq 0 \end{cases}$

A/2016, A/2011, S/2010, S/2009

CONVOLUTION of Two Function

Definition:- Let $f(t)$ and $g(t)$ be piecewise continuous functions on $[0, \infty)$. The convolution of "f" and "g" denoted by $f * g$ is defined by

$$(f * g)t = \int_0^t f(t-u)\, g(u)\, du$$

$$= \int_0^t f(u)\, g(t-u)\, du$$

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It can be easily verified that

(1) - $f * g = g * f$

(2) - $f * (g+h) = f*g + f*h$

(3) - $f * 0 = 0$

** **

LEMMA

$$f * g = \int_0^{\infty} H(t-u) f(u) g(t-u)\, du$$

Proof

using the definition of unit step function that

$$H(t-u) = \begin{cases} 1 & ; t > u \\ 0 & ; t < u \end{cases}$$

Now by Definition

$$f * g = \int_0^t f(u)\, g(t-u)\, du = \int_0^t f(u) \cdot$$

$$= \int_0^t 1 \cdot f(u) \cdot g(t-u) \cdot du$$

$$= \int_0^t H(t-u) \cdot f(u) \cdot g(t-u) \cdot du \quad (\because t > u)$$

$$f * g = \int_0^t H(t-u) f(u) g(t-u)\, du + \int_t^{\infty} H(t-u) f(u) g(t-u)\, du$$

$$f * g = \int_0^{\infty} H(t-u) f(u) \cdot g(t-u)\, du$$

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A/2016, A/2015, A/2013

CONVOLUTION THEOREM

Statment :- If $f(t)$ and $g(t)$ are piecewise continous functions over the interval $[0, \tau)$, where τ is an arbitrary positive number, and are of exponential order. Then

$$\mathcal{L}\{f * g\} = F(s)\, G(s)$$

This same result can also be stated as

$$\mathcal{L}^{-1}\{F(s)\, G(s)\} = f * g$$

where $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$

Proof :- As we know that

$$f * g = \int_0^t f(u) \cdot g(t-u) \cdot du$$

$$\mathcal{L}\{f * g\} = \int_0^\infty e^{-st} \left[\int_0^t f(u)\, g(t-u) \cdot du \right] dt$$

By using Lemma (Proved in pervious page)

$$\mathcal{L}\{f * g\} = \int_0^\infty e^{-st} \int_0^\infty H(t-u) f(u)\, g(t-u)\, du \cdot dt$$

where $H(t-u)$ is the unit step function

Integrating out "t"

$$= \int_0^\infty \left[\int_0^\infty e^{-st} H(t-u) g(t-u) \cdot dt \right] f(u) \cdot du$$

Let $t - u = \tau$

$dt = d\tau$

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when $t \to 0$ Then $\tau \to -u$

when $t \to \infty$ Then $\tau \to \infty$

$$= \int_0^\infty \left[\int_{-u}^\infty e^{-s(\tau+u)} H(\tau) g(\tau) d\tau \right] f(u) du$$

$$= \int_0^\infty \left[\int_{-u}^0 e^{-s(\tau+u)} H(\tau) g(\tau) d\tau + \int_0^\infty e^{-s(\tau+u)} H(\tau) g(\tau) d\tau \right] f(u) du$$

$$= \int_0^\infty \left[\int_0^\infty e^{-s\tau} \cdot e^{-su} \cdot H(\tau) d\tau \right] f(u) \cdot du$$

$$H(\tau) = \begin{cases} 1 ; \tau > 0 \\ 0 ; \tau < 0 \end{cases}$$

$$= \int_0^\infty e^{-su} \left[\int_0^\infty e^{-s\tau} H(\tau) g(\tau) d\tau \right] f(u) du$$

$$= \int_0^\infty e^{-su} f(u) \cdot du \left[\int_0^\infty e^{-s\tau} g(\tau) d\tau \right]$$

$$= \int_0^\infty e^{-su} f(u) \cdot du \; [G(s)]$$

$$= G(s) \int_0^\infty e^{-su} f(u) \cdot du$$

$$= G(s) \cdot F(s)$$

OR

$$= F(s) \cdot G(s)$$

Therefore

$$\mathcal{L}\{f * g\} = F(s) \cdot G(s)$$

Hence proved

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Exercise :- Q.1 = Find the inverse laplace transform using the convolution Theorem

$$\frac{4}{s^2(s-2)}$$

Solution :- $\frac{4}{s^2(s-2)} = \frac{4}{s^2} \cdot \frac{1}{s-2} = F(s) \cdot G(s)$

Here $F(s) = \frac{4}{s^2}$ and $G(s) = \frac{1}{s-2}$

$\mathcal{L}^{-1}\{F(s)\} =$

$$\mathcal{L}^{-1}\left\{\frac{4}{s^2}\right\} = 4\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = 4 \cdot t = f(t)$$

$$\mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t} = g(t)$$

By Convolution Theorem

$$\mathcal{L}^{-1}\{F(s) \cdot G(s)\} = f * g$$

$$\mathcal{L}^{-1}\left\{\frac{4}{s^2} \cdot \frac{1}{s-2}\right\} = \int_0^t f(t-u) \cdot g(u) \cdot du$$

$$= \int_0^t 4(t-u) \cdot e^{2u} \cdot du$$

$$= 4\int_0^t te^{2u} - u \cdot e^{2u} \cdot du$$

$$= 4t\left[\frac{e^{2u}}{2}\right|_0^t - 4\left[u \cdot \frac{e^{2u}}{2} - \frac{e^{2u}}{4}\right|$$

$$= \frac{4t}{2}\left[e^{2t} - 1\right] - 4\left[\frac{te^{2t}}{2} - \frac{e^{2t}}{4} - 0 + \frac{1}{4}\right.$$

$$= 2te^{2t} - 2t - 2te^{2t} + e^{2t} - 1$$

$$= e^{2t} - 2t - 1$$

Answer

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Q2 $\Rightarrow \dfrac{s}{(s^2+9)^2}$

Solution:- $\dfrac{s}{(s^2+9)^2} = \dfrac{s}{(s^2+9)} \cdot \dfrac{1}{(s^2+9)} = F(s) \cdot G(s)$

where

$F(s) = \dfrac{s}{s^2+9}$; $G(s) = \dfrac{1}{s^2+9}$

$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\dfrac{s}{s^2+9}\right\} = \cos 3t$

$g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s^2+3^2}\right\} = \dfrac{1}{3}\mathcal{L}^{-1}\left\{\dfrac{3}{s^2+3^2}\right\}$

$g(t) = \dfrac{1}{3}\sin 3t$

By Convolution Theorem

$\mathcal{L}^{-1}\{F(s) \cdot G(s)\} = f * g$

$= \int_0^t f(t-u) \cdot g(u) \cdot du$

$= \int_0^t \cos 3(t-u) \cdot \dfrac{1}{3}\sin 3u \cdot du$

$= \dfrac{1}{3}\int_0^t [\cos 3t \cdot \cos 3u + \sin 3t \cdot \sin 3u] \sin 3u \cdot du$

$= \dfrac{1}{3}\int_0^t [\cos 3t \cdot \cos 3u \cdot \sin 3u + \sin 3t \cdot \sin^2 3u]\, du$

$= \dfrac{\cos 3t}{3 \cdot 2}\int_0^t 2\cos 3u \cdot \sin 3u\, du + \dfrac{\sin 3t}{3}\int_0^t \left(\dfrac{1-\cos 6u}{2}\right) du$

$= \dfrac{\cos 3t}{6}\int_0^t \sin 6u \cdot du + \dfrac{\sin 3t}{6}\int_0^t (1-\cos 6u)\, du$

$= \dfrac{\cos 3t}{6}\left[-\dfrac{\cos 6u}{6}\right]_0^t + \dfrac{\sin 3t}{6}\left[u - \dfrac{\sin 6u}{6}\right]_0^t$

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$$= \frac{\cos 3t}{6}\left[-\frac{\cos 6t}{6} + \frac{\cos(0)}{6}\right] + \frac{\sin 3t}{6}\left[t - \frac{\sin 6t}{6} - 0\right]$$

$$= -\frac{\cos 3t \cdot \cos 6t}{36} + \frac{\cos 3t}{36} + \frac{t \sin 3t}{6} - \frac{\sin 3t \cdot \sin 6t}{36}$$

$$= -\frac{1}{36}\left[\cos 3t \cdot \cos 6t + \sin 3t \cdot \sin 6t\right] + \frac{\cos 3t}{36} + \frac{t \sin 3t}{6}$$

$$= -\frac{1}{36}\left[\cos(3t - 6t)\right] + \frac{\cos 3t}{36} + \frac{t \sin 3t}{6}$$

$$= -\frac{1}{36}\left[\cos(-3t)\right] + \frac{\cos 3t}{36} + \frac{t \sin 3t}{6}$$

$$= -\frac{\cos 3t}{36} + \frac{\cos 3t}{36} + \frac{t \sin 3t}{6}$$

$$= \frac{t \sin 3t}{6} \quad \text{Answer}$$

Q3 $\Rightarrow \dfrac{3}{s^2(s^2+9)}$

Solution:- $\dfrac{3}{s^2(s^2+9)} = \dfrac{1}{s^2} \cdot \dfrac{3}{s^2+9} = F(s) \cdot G(s)$

Here $f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$

$g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} = \sin 3t$

By Convolution Theorem

$$\mathcal{L}^{-1}\{F(s)\,G(s)\} = f * g$$

$$= \int_0^t f(t-u) \cdot g(u) \cdot du$$

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$$= \int_0^t (t-u)\sin 3u \cdot du$$

$$= \int_0^t t\sin 3u\,du - \int_0^t u\sin 3u \cdot du$$

$$= t\int_0^t \sin 3u\,du - \int_0^t u \cdot \sin 3u \cdot du$$

$$= t\left[\frac{-\cos 3u}{3}\right|_0^t - \left[u \cdot \frac{(-\cos 3u)}{3}\right|_0^t - \int_0^t -\frac{\cos 3u}{3} \cdot du$$

$$= t\left[\frac{-\cos 3t}{3} + \frac{\cos 0}{3}\right] + \left[\frac{t\cos 3t}{3} - 0\right] - \frac{1}{3}\left[\frac{\sin 3u}{3}\right|_0^t$$

$$= -\frac{t\cos 3t}{3} + \frac{t}{3} + \frac{t\cos 3t}{3} - \frac{\sin 3t}{9}$$

$$= \frac{t}{3} - \frac{\sin 3t}{9}$$

$$= \frac{1}{9}[3t - \sin 3t] \quad \text{Answer}$$

Written by
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Q4⇒ $\mathcal{L}^{-1}\left\{\frac{s}{(s+1)(s^2+4)}\right\}$

Solution:- $\frac{1}{(s+1)(s^2+4)} = \frac{1}{(s+1)} \cdot \frac{s}{s^2+4} = F(s)G(s)$

where $F(s) = \frac{1}{s+1}$ and $G(s) = \frac{s}{s^2+4}$

Then $f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$

$g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t$

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By Convolution Theorem

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f * g = \int_0^t f(t-u)g(u)\,du$$

$$= \int_0^t e^{-(t-u)} \cos 2u \cdot du$$

$$= \int_0^t e^{-t} \cdot e^{u} \cdot \cos 2u \cdot du$$

$$= e^{-t}\int_0^t e^{u} \cdot \cos 2u \cdot du \quad \text{---(1}$$

Consider $\int_0^t e^{u} \cdot \cos 2u \cdot du$

$$I = \int_0^t \cos 2u \cdot e^{u} \cdot du$$

$$= \cos 2u \cdot e^{u}\Big|_0^t - \int_0^t e^{u}(-2\sin 2u)\,du$$

$$= [e^{t} \cdot \cos 2t - 1] + 2\int_0^t \sin 2u \cdot e^{u} \cdot du$$

$$= [e^{t} \cdot \cos 2t - 1] + 2\left\{\sin 2u \cdot e^{u}\Big|_0^t - \int_0^t e^{u} 2\cos 2u\right.$$

$$= [e^{t} \cdot \cos 2t - 1] + 2\{e^{t} \cdot \sin 2t - 2\int_0^t e^{u} \cdot \cos 2u$$

$$I = [e^{t} \cdot \cos 2t - 1] + 2e^{t} \cdot \sin 2t - 4I$$

$$I + 4I = [e^{t} \cdot \cos 2t + 2e^{t} \cdot \sin 2t - 1]$$

$$5I = e^{t}[\cos 2t + 2\sin 2t - e^{-t}]$$

$$I = \frac{e^{t}}{5}[\cos 2t + 2\sin 2t - e^{-t}]$$

Putting this (1)

$$= e^{-t} \cdot \frac{e}{5}^{t}[\cos 2t + 2\sin 2t - e^{-t}]$$

$$= \frac{1}{5}[\cos 2t + 2\sin 2t - e^{-t}] \quad \text{Answer}$$

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Q5⇒ $\mathcal{L}^{-1}\left\{\frac{2K^2}{(S^2+K^2)^2}\right\}$

Solution:- $\frac{2K^2}{(S^2+K^2)^2} = \frac{2K}{(S^2+K^2)} \cdot \frac{K}{S^2+K^2} = F(S)\,G(S)$

where $F(S) = \frac{2K}{S^2+K^2}$ and $G(S) = \frac{K}{S^2+K^2}$

Therefor $f(t) = \mathcal{L}^{-1}\{F(S)\} = \mathcal{L}^{-1}\left\{\frac{2K}{S^2+K^2}\right\} = 2\sin Kt$

$g(t) = \mathcal{L}^{-1}\{G(S)\} = \mathcal{L}^{-1}\left\{\frac{K}{S^2+K^2}\right\} = \sin Kt$

By Convolution Theorem

$$\mathcal{L}^{-1}\{F(S)\cdot G(S)\} = f(t) * g(t)$$

$$= \int_0^t f(t-u)\cdot g(u)\cdot du$$

$$= \int_0^t 2\sin K(t-u)\sin Ku\,.\,du$$

$$= 2\int_0^t [\sin Kt\cdot\cos Ku - \cos Kt\cdot\sin Ku]\cdot\sin Ku\,.\,du$$

$$= 2\sin Kt\int_0^t \cos Ku\cdot\sin Ku\,du - 2\cos Kt\int_0^t \sin^2 Ku\,.\,du$$

$$= \sin Kt\int_0^t 2\cos Ku\cdot\sin Ku\,du - 2\cos Kt\int_0^t \frac{1-\cos 2Ku}{2}\,.\,du$$

$$= \sin Kt\int_0^t \sin 2Ku\,.\,du - \cos Kt\int_0^t (1-\cos 2Ku)\,du$$

$$= \sin Kt\left[-\frac{\cos 2Ku}{2K}\right]_0^t - \cos Kt\left[u - \frac{\sin 2Ku}{2K}\right]_0^t$$

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$$= \sin Kt\left[\frac{-\cos 2Kt}{2K} + \frac{\cos(0)}{2K}\right] - \cos Kt\left[t - \frac{\sin 2Kt}{2K}\right.$$

$$= \frac{-\sin Kt \cdot \cos 2Kt}{2K} + \frac{\sin Kt}{2K} - t\cos Kt + \frac{\cos Kt \cdot \sin 2K}{2K}$$

$$= \frac{1}{2K}\left[\sin 2Kt \cdot \cos Kt - \cos 2Kt \sin Kt\right] + \frac{\sin Kt}{2K} - t\cos K$$

$$= \frac{\sin(2Kt - Kt)}{2K} + \frac{\sin Kt}{2K} - t\cos Kt$$

$$= \frac{\sin Kt}{2K} + \frac{\sin Kt}{2K} - t\cos Kt$$

$$= \frac{\sin Kt}{K} - t\cos Kt \quad \text{Answer}$$

Example:- $\mathcal{L}^{-1}\left\{\frac{3s+17}{s^2+8s+25}\right\}$

Solution: $\frac{3s+17}{s^2+8s+25} = \frac{3s+12+5}{s^2+8s+16+9} = \frac{3(s+4)+5}{(s+4)^2+3^2}$

$$\frac{3(s+4)+5}{(s+4)^2+3^2} = \frac{3(s+4)}{(s+4)^2+3^2} + \frac{5}{(s+4)^2+3^2}$$

Taking Inverse Laplace

$$\mathcal{L}^{-1}\left\{\frac{3s+17}{s^2+8s+25}\right\} = 3\mathcal{L}^{-1}\left\{\frac{s+4}{(s+4)^2+3^2}\right\} + 5\mathcal{L}^{-1}\left\{\frac{1}{(s+4)^2+3^2}\right.$$

$$\mathcal{L}\{e^{Kt}\cos \omega t\} = \frac{s-K}{(s-K)^2+\omega^2} \quad \& \quad \mathcal{L}\{e^{Kt}\sin \omega t\} = \frac{\omega}{(s-K)^2+\omega^2}$$

$$= 3e^{-4t}\cos 3t + \frac{5}{3}e^{-4t}\sin 3t$$

Answer.

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A/2014, S/2010, A/2008

SECOND SHIFTING THEOREM

It is also called second translation Theorem

Statment:- Let f be a function of exponential order "a" and $\mathcal{L}\{f(t)\} = F(s)$

Then $$\mathcal{L}^{-1}\{e^{-as}F(s)\} = H(t-a)\,f(t-a)$$

Proof:- let $f(t) = H(t-a)\,f(t-a)$

By Definition of Laplace Transform

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$\mathcal{L}\{H(t-a)f(t-a)\} = \int_0^\infty e^{-st} H(t-a)f(t-a)\,dt$$

$$= \int_0^a e^{-st} H(t-a)f(t-a)\,dt + \int_a^\infty e^{-st} H(t-a)f(t-a)\,dt$$

(here $H(t-a) = 0$ on $0 \to a$, and $H(t-a) = 1$ on $a \to \infty$)

As we know $H(t-a) = \begin{cases} 1 & ; t > a \\ 0 & ; t < a \end{cases}$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

put $t - a = \theta$

$dt = d\theta$

$t \to a$ Then $\theta \to 0$

$t \to \infty$ Then $\theta \to \infty$

$$\mathcal{L}\{H(t-a)f(t-a)\} = \int_0^\infty e^{-s(a+\theta)} f(\theta)\,d\theta$$

$$= \int_0^\infty e^{-sa-s\theta} \cdot f(\theta)\,d\theta$$

$$= \int_0^\infty e^{-sa} \cdot e^{-s\theta} \cdot f(\theta)\,d\theta$$

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$$= e^{-as}\int_0^\infty e^{-st} f(t)\cdot dt$$

$$\mathcal{L}\{H(t-a)f(t-a)\} = e^{-as}F(s)$$

OR $$\mathcal{L}^{-1}\{e^{-as}F(s)\} = H(t-a)f(t-a)$$

A/2014

Question :- $\mathcal{L}^{-1}\left\{\frac{e^{-as}}{(s+1)^3}\right\}$

Solution :- Here $F(s) = \frac{1}{(s+1)^3}$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\} \quad \text{---(i)}$$

As we know that

$$\mathcal{L}\{t^n e^{kt}\} = \frac{n!}{(s-k)^{n+1}}$$

Put $n=2$; $k=-1$

$$\mathcal{L}\{t^2 e^{-t}\} = \frac{2!}{(s+1)^3}$$

$$t^2\cdot e^{-t} = \mathcal{L}^{-1}\left\{\frac{2!}{(s+1)^3}\right\}$$

$$t^2\cdot e^{-t} = 2!\,\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\}$$

$$\frac{t^2\cdot e^{-t}}{2!} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\}$$

put this in (i)

$$f(t) = \frac{1}{2!}t^2\cdot e^{-t}$$

BY second shifting theorem

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = H(t-a)\,f(t-a)$$

There for $\mathcal{L}^{-1}\left\{e^{-as}\frac{1}{(s+1)^3}\right\} = H(t-a)\cdot\frac{e^{-(t-a)}}{2!}(t-a)^2$

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A/2010

Question :- $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{(s+1)^3}\right\}$

Solution :- Here $f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\}$

$$f(t) = \frac{1}{2!} t^2 e^{-t}$$

Using 2nd Shifting Theorem, we get

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{(s+1)^3}\right\} = H(t-2) f(t-2) \quad \ldots$$

$$= H(t-2) \cdot \frac{1}{2!} (t-2)^2 e^{-(t-2)}$$

$$= H(t-2) \cdot \frac{1}{2!} (t-2)^2 e^{(2-t)}$$

A/2009

Question :- $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s+2)^4}\right\}$

Solution :- Here $F(s) = \frac{1}{(s+2)^4}$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^4}\right\}$$

Using Formula $\mathcal{L}\{t^n e^{Kt}\} = \frac{n!}{(s-K)^{n+1}}$

put $n = 3$ and $K = -2$

$$\mathcal{L}\{t^3 e^{-2t}\} = \frac{3!}{(s+2)^4}$$

$$t^3 \cdot e^{-2t} = \mathcal{L}^{-1}\left\{\frac{3!}{(s+2)^4}\right\}$$

$$t^3 \cdot e^{-2t} = 3!\, \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^4}\right\}$$

$$\frac{1}{3!} t^3 e^{-2t} = \mathcal{L}^{-1}\left\{1/(s+2)^4\right\}$$

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Therefor $f(t) = \frac{1}{3!} t^3 e^{-2t}$

Using 2nd Shifting Theorem

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s+2)^4}\right\} = H(t-1) f(t-1)$$

$$= H(t-1) \frac{1}{3!} (t-1)^3 e^{-2(t-1)}$$

A/2008

Question :- $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s^2+1)^2}\right\}$

Solution :- Here $F(s) = \frac{1}{(s^2+1)^2}$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$$

we find inverse By Convolation Theor

$$f(t) = \frac{1}{2}[\sin t - t\cos t]$$

using 2nd Shifting Theorem

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s^2+1)^2}\right\} = H(t-1).f(t-1)$$

$$= H(t-1) \frac{1}{2}[\sin(t-1) - (t-1)\cos(t-1)]$$

Question :- $\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}$

Solution :-

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$$

using 2nd Shifting Theorem

$$\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\} = H(t-\pi) \cdot f(t-\pi) =$$

$$= H(t-\pi) \cdot \sin(t-\pi) = H(t-\pi)(-\sin t)$$

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Solutions of INITIAL VALUE PROBLEMS

** ═══════════ **

Question 1 :- Use the Laplace transform to Solve $u' - 2u = 0$, $u(0) = 1$

Solution :- Given that

$u' - 2u = 0$; $u(0) = 1$

Taking Laplace transform on both side

$$\mathcal{L}\{u' - 2u\} = 0$$

$$\mathcal{L}\{u'\} - 2\mathcal{L}\{u\} = 0$$

$$[sU(s) - u(0)] - 2U(s) = 0$$

$$sU(s) - 2U(s) - u(0) = 0$$

$$[s-2]U(s) - 1 = 0$$

$$(s-2)U(s) = 1$$

$$U(s) = \frac{1}{s-2}$$

Taking Laplace inverse on both side

$$\mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$u(t) = e^{2t}$$

Hence $u(t) = e^{2t}$ is the solution of given I.V.P

** ═══════════ **

Question 2 :- $u'' - 2u' + u = te^t$

$u(0) = 0$, $u'(0) = 0$

Solution :- Given that

$$u'' - 2u' + u = te^t$$

Taking Laplace transform on both side

$$\mathcal{L}\{u'' - 2u' + u\} = \mathcal{L}\{te^t\}$$

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Question 4:- $t u'' - t u' - u = 0$

$u(0) = 0 \quad , \quad u'(0) = 3$

Solution:- Given that

$$t v'' - t v' - v = 0$$

Taking laplace Transform on both side

$$\mathcal{L}\{t v''\} - \mathcal{L}\{t v'\} - \mathcal{L}\{v\} = 0$$

$$(-1)^1 \frac{d}{ds}\{\mathcal{L}(u'')\} - (-1)^1 \frac{d}{ds}\{\mathcal{L}(u')\} - v(s) = 0$$

$$-\frac{d}{ds}\{s^2 v(s) - s\,v(0) - v'(0)\} + \frac{d}{ds}\{s\,v(s) - v(0)\} - v(s)$$

given $v(0) = 0$ & $v'(0) = 3$

$$-\frac{d}{ds}\{s^2 v(s) - 0 - 3\} + \frac{d}{ds}\{s\,v(s) - 0\} - v(s) = 0$$

$$-\{s^2 v'(s) + v(s)(2s) - 0\} + \{s\,v'(s) + v(s)\} - v(s) =$$

$$-s^2 v'(s) - 2s\,v(s) + s\,v'(s) + v(s) - v(s) = 0$$

$$(-s^2 + s)\,v'(s) - 2s\,v(s) = 0$$

$$s(1-s)\,v'(s) = 2s\,v(s)$$

$$\frac{v'(s)}{v(s)} = \frac{2}{1-s} = \frac{-2}{s-1}$$

Integrating both side

$$\int \frac{v'(s)}{v(s)} = -2 \int \frac{1}{s-1}\, ds$$

$$\ln|v(s)| = -2 \ln|(s-1)| + \ln G$$

$$\ln|v(s)| = -\ln[s-1]^2 + \ln G$$

$$\ln v(s) + \ln(s-1)^2 = \ln G$$

$$\ln[v(s) \cdot (s-1)^2] = \ln G$$

$$v(s)(s-1)^2 = G$$

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$$U(s) = \frac{c}{(s-1)^2}$$

Taking Laplace inverse on both side

$$\mathcal{L}^{-1}\{U(s)\} = c\,\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$U(t) = c\,[te^t]$$

Now Differentiate w.r.t "t"

$$U'(t) = c\,[te^t + e^t]$$

Given $U'(0) = 3 \Rightarrow U'(0) = c[0 \cdot e^0 + e^0]$

$= c[0+1]$

$= c = 3$

Hence $U(t) = 3te^t$

A/2015

Question 5 :- Solve the I.V.P

$$y''(t) + t\,y'(t) - y(t) = 0$$

$$y(0) = 0, \quad y'(0) = 1$$

Solution :- Given that

$$y''(t) + t\,y'(t) - y(t) = 0$$

Taking Laplace transform on both side

$$\mathcal{L}\{y''(t)\} + \mathcal{L}\{t\,y'(t)\} - \mathcal{L}\{y(t)\} = 0$$

$$[s^2 y(s) - sy(0) - y'(0)] + (-1)^1 \frac{d}{ds}[\mathcal{L}\{y'(t)\} - y(s)] = 0$$

$$s^2 y(s) - sy(0) - y'(0)] - \frac{d}{ds}[s\,y(s) - y(0)] - y(s) = 0$$

$$s^2 y(s) - 0 - 1] - \frac{d}{ds}[s\,y(s) = 0] - y(s) = 0$$

$$s^2 y(s) - 1] - [s\,y'(s) + y(s)] - y(s) = 0$$

$$s^2 y(s) - 1 - sy'(s) - y(s) - y(s) = 0$$

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$$\mathcal{L}\{v''\} - 2\mathcal{L}\{v'\} + \mathcal{L}\{v\} = \mathcal{L}\{te^t\}$$

$$[s^2V(s) - s\,v(0) - v'(0)] - 2[sV(s) - v(0)] + V(s) = \frac{1}{(s-1)}$$

using given conditions $v(0) = v'(0) = 0$

$$[s^2V(s) - 0 - 0] - 2[sV(s) - 0] + V(s) = \frac{1}{(s-1)^2}$$

$$[s^2 - 2s + 1]\,V(s) = \frac{1}{(s-1)^2}$$

$$(s-1)^2 V(s) = \frac{1}{(s-1)^2}$$

$$V(s) = \frac{1}{(s-1)^4}$$

Taking Laplace inverse on both side

$$\mathcal{L}^{-1}\{V(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^4}\right\}$$

$$v(t) = \frac{1}{3!}t^3e^t \quad \text{required solution}$$

++ ++

Question 3:- $v'' + 4v' + 3v = 0$ $\quad v(0) = 1$, $v'(0) = 0$

Solution:- Given that

$$v'' + 4v' + 3v = 0$$

Taking Laplace Transform on both sides

$$\mathcal{L}\{v''\} + 4\mathcal{L}\{v'\} + 3\mathcal{L}\{v\} = 0$$

$$\{s^2V(s) - sv(0) - v'(0)\} + 4\{sV(s) - v(0)\} + 3V(s) = 0$$

$$[s^2V(s) - s(1) - 0] + 4[sV(s) - 1] + 3V(s) = 0$$

$$s^2V(s) - s + 4sV(s) - 4 + 3V(s) = 0$$

$$[s^2 + 4s + 3]V(s) = s + 4$$

$$V(s) = \frac{s+4}{(s^2+4s+3)}$$

$$\therefore = \frac{s+4}{s^2+3s+s+3} = \frac{s+4}{s(s+3)+1(s+3)}$$

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$$U(S) = \frac{(S+4)}{(S+1)(S+3)} \quad \text{——— (1)}$$

$$\frac{S+4}{(S+1)(S+3)} = \frac{A}{S+1} + \frac{B}{S+3} \quad \text{——— (ii)}$$

Multiplying both sicle by $(S+1)(S+3)$

$$(S+4) = A(S+3) + B(S+1) \quad \text{—— (iii)}$$

Put $S = -3$ in (iii)

$$-3+4 = A(-3+3) + B(-3+1)$$

$$1 = 0 - 2B \Rightarrow \boxed{B = -1/2}$$

Now put $S = -1$ in (iii)

$$-1+4 = A(-1+3) + B(-1+1)$$

$$3 = 2A + 0 \Rightarrow \boxed{A = 3/2}$$

putting the values of A and B in (ii)

$$\frac{S+4}{(S+1)(S+3)} = \frac{3}{2(S+1)} - \frac{1}{2(S+3)}$$

There for (i) becomes

$$U(S) = \frac{3}{2(S+1)} - \frac{1}{2(S+3)}$$

Taking Laplace inverse Transform on both side

$$\mathcal{L}^{-1}\{U(S)\} = \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{S+1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{S+3}\right\}$$

$$= \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}$$

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LaEab

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$$(s^2-2)y(s) - 1 - s\,y'(s) = 0$$

$$s\,y'(s) + (2-s^2)y(s) + 1 = 0$$

$$y'(s) + \left[\frac{2-s^2}{s}\right] y(s) + \frac{1}{s} = 0 \quad \text{——(1)}$$

which is the form of $y' + P(x)y = Q(x)$

And in Equation (1) Integration factor is

$$I.F = \int \frac{2-s^2}{s}\,ds$$

$$= e^{\int (\frac{2}{s} - s)\,ds} \qquad = e^{2\int \frac{1}{s}\,ds - \int s\,ds}$$

$$= e^{2\ln|s| - s^2/2}$$

$$= e^{\ln s^2 - s^2/2}$$

$$= e^{\ln s^2} \cdot e^{-s^2/2}$$

$$= s^2 e^{-s^2/2}$$

Multiply (1) by I.F, we get

$$\underbrace{s^2 e^{-s^2/2} y'(s) + s^2 e^{-s^2/2}\left[\frac{2-s^2}{s}\right] y(s)} + \frac{s^2 e^{-s^2/2}}{s} = 0$$

$$\frac{d}{ds}\left[s^2 e^{-s^2/2} y(s)\right] + s e^{-s^2/2}$$

$$\frac{d}{ds}\left[s^2 e^{-s^2/2} y(s)\right] = -s e^{-s^2/2}$$

integrating both side

$$s^2 e^{-s^2/2} y(s) = -\int s e^{-\frac{s^2}{2}}\,ds$$

Put $\frac{s^2}{2} = \theta$

$$\frac{2s\,ds}{2} = d\theta$$

$$s^2 e^{-s^2/2} y(s) = -\int e^{-\theta}\,d\theta$$

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$$s^2 e^{-s^2/2} y(s) = -\left[\frac{e^{-0}}{-1}\right] + c = e^{-0} + c$$

$$s^2 e^{-s^2/2} y(s) = e^{-s^2/2} + c \quad \text{——— (2)}$$

$$y(s) = \frac{e^{-s^2/2}}{s^2 e^{-s^2/2}} + \frac{c}{s^2 e^{-s^2/2}}$$

$$y(s) = \frac{1}{s^2} + \frac{c}{s^2 e^{-s^2/2}} \quad \text{——— (3)}$$

Applying Limit on (2) for finding c

$$\lim_{s\to\infty} \left[s^2 e^{-s^2/2} y(s)\right] = \lim_{s\to\infty} \left[e^{-s^2/2} + c\right]$$

$$0 = 0 + c$$

$$\boxed{c = 0}$$

Put in (3) $\Rightarrow$ $y(s) = \frac{1}{s^2}$

Now Taking Laplace inverse

$$\mathcal{L}^{-1}\{y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$\boxed{y(t) = t}$$

1/2013

Question 6 :- Solve I.V.P by using laplace transform

$$y'' + 4y' + 6y = 1 + e^{-t}$$

$$y(0) = 0 = y'(0)$$

Solution :- Given that

$$y'' + 4y' + 6y = 1 + e^{-t}$$

Taking laplace Transform on both side

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$$\mathcal{L}\{y''(t)\} + 4\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} = \mathcal{L}\{1\} + \mathcal{L}\{e^{-t}\}$$

$$[s^2Y(s) - sY(0) - Y'(0)] + 4[sY(s) - y(0)] + 6Y(s) = \frac{1}{s} + \frac{1}{s+1}$$

$$[s^2Y(s) - 0 - 0] + 4[sY(s) - 0] + 6\cdot Y(s) = \frac{1}{s} + \frac{1}{s+1}$$

$$[s^2 + 4s + 6]\,Y(s) = \frac{1}{s} + \frac{1}{s+1}$$

$$[s^2 + 4s + 4 + 2]\,Y(s) = \frac{1}{s} + \frac{1}{s+1}$$

$$[(s+2)^2 + 2]\,Y(s) = \frac{1}{s} + \frac{1}{s+1}$$

$$Y(s) = \frac{1}{s[(s+2)^2+2]} + \frac{1}{(s+1)[(s+2)^2+2]}$$

Now Taking Laplace inverse

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s[(s+2)^2+2]}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)[(s+2)^2+2]}\right\}$$

$\downarrow$ **A** $\qquad\qquad$ $\downarrow$ **B**

Now we find Laplace inverse by using Convolution Theorem

Consider $\mathbf{A} = \mathcal{L}^{-1}\left\{\frac{1}{s[(s+2)^2+2]}\right\}$

Here $F(s) = \frac{1}{s}$ and $G(s) = \frac{1}{(s+2)^2 + (\sqrt{2})^2}$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2 + (\sqrt{2})^2}\right\} = \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s+2)^2 + (\sqrt{2})^2}\right\}$$

$$= \frac{1}{\sqrt{2}} e^{-2t}\sin\sqrt{2}t$$

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By Convolution Theorem

$$\mathcal{L}^{-1}[F(s)\cdot G(s)] = f*g = \int_0^t f(t-u)\cdot g(u)\cdot du$$

Let

$$I_1 = \int_0^t 1\cdot e^{-2u}\cdot \sin\sqrt{2}u\cdot du$$

$$I_1 = \int_0^t \sin\sqrt{2}u\cdot e^{-2u}\cdot du$$

$$= \sin\sqrt{2}u\cdot\frac{e^{-2u}}{-2}\Big|_0^t - \int_0^t \sqrt{2}\cos\sqrt{2}u\cdot\frac{e^{-2u}}{-2}\,du$$

$$= \frac{-e^{-2t}}{2}\sin\sqrt{2}t + \frac{\sqrt{2}}{2}\int_0^t \cos\sqrt{2}u\, e^{-2u}\,du$$

$$= -\frac{e^{-2t}}{2}\sin\sqrt{2}t + \frac{\sqrt{2}}{2}\left[\cos\sqrt{2}u\,\frac{e^{-2u}}{-2}\Big|_0^t - \int_0^t \frac{e^{-2u}}{-2}\cdot -\sqrt{2}\sin\sqrt{2}u\,du\right]$$

$$= -\frac{e^{-2t}}{2}\sin\sqrt{2}t - \frac{\sqrt{2}}{4}\left[e^{-2t}\cos\sqrt{2}t - 1\right] - \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}\int_0^t e^{-2u}\sin\sqrt{2}u\,du$$

$$I_1 = -\frac{e^{-2t}}{2}\sin\sqrt{2}t - \frac{1}{2\sqrt{2}}\left[e^{-2t}\cos\sqrt{2}t - 1\right] - \frac{1}{2}I_1$$

$$I_1 + \frac{1}{2}I_1 = \frac{-e^{-2t}}{2}\sin\sqrt{2}t - \frac{1}{2\sqrt{2}}\left[e^{-2t}\cos\sqrt{2}t - 1\right]$$

$$\frac{3}{2}I_1 = \frac{1}{2}\left[-e^{-2t}\sin\sqrt{2}t - \frac{1}{\sqrt{2}}\left\{e^{-2t}\cos\sqrt{2}t - 1\right\}\right]$$

$$I_1 = \frac{1}{3}\left[e^{-2t}\sin\sqrt{2}t - \frac{1}{\sqrt{2}}\left\{e^{-2t}\cos\sqrt{2}t - 1\right\}\right]$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s\left((s+2)^2+(\sqrt{2})^2\right)}\right\} = \frac{1}{\sqrt{2}}\left[\frac{1}{3}\left[e^{-2t}\sin\sqrt{2}t - \frac{1}{\sqrt{2}}\left\{e^{-2t}\cos\sqrt{2}t - 1\right\}\right]\right]$$

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Now Take "B" and we find inverse by Convolution Theorem

$$B = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)[(s+2)^2+2]}\right\}$$

$$= \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s+1)[(s+2)^2+(\sqrt{2})^2]}\right\}$$

Here $F(s) = \frac{1}{s+1}$ and $G(s) = \frac{\sqrt{2}}{(s+2)^2+(\sqrt{2})^2}$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$$

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s+2)^2+(\sqrt{2})^2}\right\} = e^{-2t}\sin\sqrt{2}$$

By Convolution Theorem

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f*g = \int_0^t f(t-u)\cdot g(u)\,du$$

$$= \int_0^t e^{-(t-u)}\cdot e^{-2u}\sin\sqrt{2}u\cdot du$$

$$= \int_0^t e^{-t}e^{u}\cdot e^{-2u}\cdot\sin\sqrt{2}u\cdot du$$

$$= e^{-t}\int_0^t e^{-u}\cdot\sin\sqrt{2}u\cdot du \quad \text{—— (i)}$$

Let $I_2 = \int_0^t \sin\sqrt{2}u\cdot e^{-u}\cdot du$

$$I_2 = \sin\sqrt{2}u(-e^{-u})\Big|_0^t - \int_0^t \sqrt{2}\cos\sqrt{2}u\,(-e^{-u})\cdot du$$

$$= -e^{-t}\sin\sqrt{2}t + \sqrt{2}\int_0^t \cos\sqrt{2}u\,e^{-u}\cdot du$$

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$$I_2 = -e^{-t}\sin\sqrt{2}t + \sqrt{2}\left[\cos\sqrt{2}\,u(-e^{-u})\Big|_0^t - \int_0^t -\sqrt{2}\sin\sqrt{2}u(-e^{-u})du\right]$$

$$I_2 = -e^{-t}\sin\sqrt{2}t + \sqrt{2}\left\{\left[-e^{-t}\cos\sqrt{2}t + 1\right] - \sqrt{2}I_2\right\}$$

$$I_2 = -e^{-t}\sin\sqrt{2}t + \sqrt{2}\left[1 - e^{-t}\cos\sqrt{2}t\right] - 2I_2$$

$$I_2 + 2I_2 = -e^{-t}\sin\sqrt{2}t + \sqrt{2}\left[1 - e^{-t}\cos\sqrt{2}t\right]$$

$$3I_2 =$$

$$I_2 = \frac{1}{3}\left[-e^{-t}\sin\sqrt{2}t + \sqrt{2}\left[1 - e^{-t}\cos\sqrt{2}t\right]\right]$$

put in (i)

$$\mathbf{B} = \frac{e^{-t}}{3\sqrt{2}}\left[-e^{-t}\sin\sqrt{2}t + \sqrt{2}\left\{1 - e^{-t}\cos\sqrt{2}t\right\}\right]$$

putting the value of A and B

$$y(t) = \frac{1}{3\sqrt{2}}\left[e^{-2t}\sin\sqrt{2}t\right] - \frac{1}{6}\left[e^{-2t}\cos\sqrt{2}t - 1\right]$$

$$\left[\frac{-e^{-2t}}{3\sqrt{2}}\sin\sqrt{2}t\right] + \frac{e^{-t}}{3}\left[1 - e^{-t}\cos\sqrt{2}t\right]$$

$$y(t) = -\frac{e^{-2t}}{6}\cos\sqrt{2}t + \frac{1}{6} + \frac{e^{-t}}{3} - \frac{e^{-2t}}{3}\cos\sqrt{2}t$$

$$= -e^{-2t}\cos\sqrt{2}t\left[\frac{1}{6} + \frac{1}{3}\right] + \frac{e^{-t}}{3} + \frac{1}{6}$$

$$y(t) = -\frac{e^{-2t}}{2}\cos\sqrt{2}t + \frac{e^{-t}}{3} + \frac{1}{6}$$

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Solutions of PARTIAL DIFFERETIAL EQUATIO

*** ***

A/2016

Question 1 :- Solve the problem using the Laplace Transform method

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} - g \qquad -\infty < x < \infty \;,\; t \geq 0$$

$$u(x,0) = u_t(x,0) = 0$$

$$u(0,t) = 0 \;, \quad \lim_{x \to \infty} u_x(x,t) = 0$$

Solution :- Given that

$$U_{tt}(x,t) = a^2 U_{xx}(x,t) - g \quad \text{---} (1)$$

Taking Laplace transform on both side

$$\mathcal{L}\{U_{tt}(x,t)\} = a^2 \mathcal{L}\{U_{xx}(x,t)\} - \mathcal{L}\{g\}$$

$$s^2 U(x,s) - sU(x,0) - U_t(x,0) = a^2 U_{xx}(x,s) - g \cdot \frac{1}{s}$$

$$s^2 U(x,s) - 0 - 0 = a^2 U_{xx}(x,s) - \frac{g}{s}$$

$$a^2 U_{xx}(x,s) - s^2 U(x,s) = \frac{g}{s}$$

$$U_{xx}(x,s) - \frac{s^2}{a^2} U(x,s) = \frac{g}{a^2 s} \quad \text{---} (2)$$

which is a non-homogenous second order D-E . Its solution can be written as

$$U(x,s) = U_c + U_p \quad \text{---} (3)$$

For Complementry function "U_c", consider D-E

$$U_{xx}(x,s) - \frac{s^2}{a^2} U(x,s) = 0$$

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$$\left[D^2 - \frac{S^2}{a^2}\right] U(x,S) = 0$$

The auxiliry equation is

$$m^2 - \frac{S^2}{a^2} = 0 \Rightarrow m = \pm S/a$$

Therefor $U_c = C_1 e^{S/a\,x} + C_2 e^{-S/a\,x}$

Now we find particular integral "U_p"

$$U_p = \frac{1}{\left[\frac{D^2 - S^2}{a^2}\right]} \cdot g/a^2S$$

$$= \frac{g}{a^2S} \cdot \frac{1}{\left[\frac{D^2 - S^2}{a^2}\right]} \cdot e^{0x}$$

$$= \frac{g}{a^2S} \cdot \frac{1}{\left[\frac{0 - S^2}{a^2}\right]} \quad (1)$$

$$U_p = \frac{g}{a^2S} \cdot \left[\frac{-a^2}{S^2}\right] = \frac{-g}{S^3}$$

Hence from ③, we get

$$U(x,S) = C_1 e^{\frac{S}{a}x} + C_2 e^{-\frac{S}{a}x} - \frac{g}{S^3} \quad \text{---} ④$$

From given conditions

$U(0,t) = 0$ and $\lim_{x \to \infty} U_x(x,t) = 0$

$\mathcal{L}\{U(0,t)\} = 0$ // $\lim_{x \to \infty} U_x(x,S) = 0$

$U(0,S) = 0$

So Equation ④ becomes

$$U(0,S) = C_1 + C_2 - \frac{g}{S^3}$$

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$$0 = C_1 + C_2 - \frac{g}{s^3} \quad \text{——(5)}$$

Now Equation (4) Diff w.r.t "x", we ha

$$U_x(x,s) = \frac{s}{a} C_1 e^{\frac{s}{a}x} - \frac{s}{a} C_2 e^{-s/a\,x}$$

$$\lim_{x\to\infty} U_x(x,s) = \frac{s}{a} C_1(\infty) - \frac{s}{a} C_2 e^{-\infty}$$

$$= \frac{s}{a} C_1(\infty) - 0 \quad = 0$$

So $\boxed{C_1 = 0}$

Put $C_1 = 0$ in (5)

$$0 = 0 + C_2 - g/s^3$$

$$\boxed{C_2 = \frac{g}{s^3}}$$

Hence

$$U(x,s) = \frac{g}{s^3} e^{-\frac{s}{a}x} - \frac{g}{s^3}$$

Taking Laplac inverse on both side

$$\mathcal{L}^{-1}\{U(x,s)\} = \mathcal{L}^{-1}\left\{\frac{g e^{-\frac{s}{a}x}}{s^3} - \frac{g}{s^3}\right\}$$

$$= g\mathcal{L}^{-1}\left\{\frac{e^{-\frac{x}{a}\cdot s}}{s^3}\right\} - g\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$$

$$U(x,t) = g\left\{\frac{1}{2}\left[t - \frac{x}{a}\right]^2 H\left[t - \frac{x}{a}\right] - \frac{t^2}{2}\right\}$$

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A/2012

Question 2:- Use Laplace Transform method to solve the problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < a, \quad 0 \le t < \infty \text{ —— (1)}$$

$$U(0,t) = 1, \quad U(1,t) = 1, \quad t > 0 \text{ —— (2)}$$

$$U(x,0) = 1 + \sin \pi x, \quad 0 \le x \le 1 \text{ —— (3)}$$

Solution:- It is one Dimensional heat problem describing conduction of heat through a rantained at zero temprature.

Now Given that $U_{xx} = U_t$

Taking Laplace transform on both sides

$$\mathcal{L}\{U_{xx}\} = \mathcal{L}\{U_t\}$$

$$U_{xx}(x,s) = s\,U(x,s) - U(x,0)$$

$$= s\,U(x,s) - 1 - \sin \pi x$$

$$\Rightarrow U_{xx}(x,s) - s\,U(x,s) = -(1 + \sin \pi x)$$

which is a non-homogeneous linear second order D-E, whose solution is given by

$$U(x,s) = U_c + U_p \text{ —— (4)}$$

First we find complementory solution

$$(D^2 - s)\,U(x,s) = 0$$

The characteristic Equation is

$$m^2 - s = 0$$

$$m = \pm\sqrt{s}$$

$$U_c = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x}$$

Now for particular Integral

$$U_p = \frac{-1}{D^2 - s}(1 + \sin \pi x)$$

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$$U_p = \frac{-1}{D^2-S} - \frac{1}{D^2-S} \sin \pi x$$

$$= \frac{-e^{0x}}{D^2-S} - \frac{1}{D^2-S} \sin \pi x$$

$$= \frac{-1}{0-S} - \frac{1}{-\pi^2-S} \sin \pi x$$

$$U_p = \frac{1}{S} + \frac{\sin \pi x}{\pi^2+S}$$

Equation (4) becomes

$$U(x,S) = C_1 e^{\sqrt{S}x} + C_2 e^{-\sqrt{S}x} + \frac{1}{S} + \frac{\sin \pi x}{\pi^2+S} \quad \text{---(5)}$$

From (2), we have

$U(0,t) = 1$, $U(1,t) = 1$

$\mathcal{L}\{U(0,t)\} = \mathcal{L}\{1\}$, $\mathcal{L}\{U(1,t)\} = \mathcal{L}\{1\}$

$U(0,S) = \frac{1}{S}$ ---(6) , $U(1,S) = 1/S$ ---(7)

From (5), on putting $x = 0$, we have

$$U(0,S) = C_1 + C_2 + \frac{1}{S} + 0$$

$$\frac{1}{S} = C_1 + C_2 + \frac{1}{S}$$

$$C_1 + C_2 = 0 \quad \text{---(8)}$$

From (5), on putting $x = 1$, we have

$$U(1,S) = C_1 e^{\sqrt{S}} + C_2 e^{-\sqrt{S}} + \frac{1}{S} + 0$$

$$\frac{1}{S} = C_1 e^{\sqrt{S}} + C_2 e^{-\sqrt{S}} + \frac{1}{S}$$

$$C_1 e^{\sqrt{S}} + C_2 e^{-\sqrt{S}} = 0 \quad \text{---(9)}$$

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From (8) and (9), we get

$$c_1 = 0 = c_2$$

These for $U(x,s) = \frac{1}{s} + \frac{1}{\pi^2+s} \sin \pi x$

Now Taking Laplace inverse on both side

$$\mathcal{L}^{-1}\{U(x,s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \sin \pi x \, \mathcal{L}^{-1}\left\{\frac{1}{s+\pi^2}\right\}$$

$$U(x,t) = 1 + \sin \pi x \cdot e^{-\pi^2 t}$$

$$= 1 + e^{-\pi^2 t} \sin \pi x$$

Required Solution

A/2010

Question 3 :- $U_{xx} = U_{tt}$, $0 < x < 1$, $t > 0$ — (1)

$U(0,t) = 0$, $U(1,t) = 0$ — (2)

$U(x,0) = \sin \pi x$, $U_t(x,0) = -\sin \pi x$ — (3)

Solution :- Taking Laplace of equ (1)

$$\mathcal{L}\{U_{xx}\} = \mathcal{L}\{U_{tt}\}$$

$$U_{xx}(x,s) = s^2 U(x,s) - sU(x,0) - U_t(x,0)$$

$$U_{xx}(x,s) = s^2 U(x,s) - s \cdot \sin \pi x + \sin \pi x$$

$$U_{xx}(x,s) - s^2 U(x,s) = -s \sin \pi x + \sin \pi x$$

which is a homogeneous second order D-E

Its solution can be written as :-

$$U(x,s) = U_c + U_p$$

First we find complementry solution

$$(D^2 - s^2) U(x,s) = 0 \qquad D = \frac{d}{dx}$$

Its characteristic equation is

$$m^2 - s^2 = 0$$

$$m = \pm s$$

$$U_c = c_1 e^{sx} + c_2 e^{-sx}$$

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And the Particular Integral U_p is

$$U_p = \frac{1}{(D^2 - s^2)} (-s \sin \pi x + \sin \pi x)$$

$$= \frac{(1-s) \sin \pi x}{D^2 - s^2}$$

$$= \frac{(1-s) \sin \pi x}{-\pi^2 - s^2} = \frac{(1-s) \sin \pi x}{-[\pi^2 + s^2]}$$

$$= \frac{(s-1) \sin \pi x}{\pi^2 + s^2}$$

As $U(x,s) = U_c + U_p$

$$= c_1 e^{sx} + c_2 e^{-sx} + \frac{(s-1) \sin \pi x}{\pi^2 + s^2} \quad \text{---} (4)$$

From (2), we have

$U(0,t) = 0$	$U(1,t) = 0$
$\mathcal{L}\{U(0,t)\} = 0$	$\mathcal{L}\{U(1,t)\} = 0$
$U(0,s) = 0$	$U(1,s) = 0$

And put $x = 0$ in (4), we get

$$U(0,s) = c_1 + c_2 + 0$$

$$0 = c_1 + c_2 \quad \text{---} (5)$$

Put $x = 1$ in (4), we get

$$U(1,s) = c_1 e^{s} + c_2 e^{-s} + 0$$

$$0 = c_1 e^{s} + c_2 e^{-s} \quad \text{---} (6)$$

Solving (5) and (6), we get

$$c_1 e^{s} - c_1 e^{-s} = 0$$

$$c_1 [e^{s} - e^{-s}] = 0$$

Since $e^{s} - e^{-s} \neq 0$ There for $c_1 = 0$

So $c_2 = 0$

Hence (4) becomes

$$U(x,s) = \frac{(s-1) \sin \pi x}{\pi^2 + s^2}$$

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Now Taking Laplace inverse on both side

$$\mathcal{L}^{-1}\{U(x,s)\} = \mathcal{L}^{-1}\left\{\frac{(s-1)\sin\pi x}{\pi^2+s^2}\right\}$$

$$U(x,t) = \mathcal{L}^{-1}\left\{\frac{s\cdot\sin\pi x}{\pi^2+s^2} - \frac{\sin\pi x}{\pi^2+s^2}\right\}$$

$$// \quad = \mathcal{L}^{-1}\left\{\frac{s\cdot\sin\pi x}{\pi^2+s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{\sin\pi x}{\pi^2+s^2}\right\}$$

$$// \quad = \sin\pi x\,\mathcal{L}^{-1}\left\{\frac{s}{\pi^2+s^2}\right\} - \frac{\sin\pi x}{\pi}\mathcal{L}^{-1}\left\{\frac{\pi}{\pi^2+s^2}\right\}$$

$$// \quad = \sin\pi x\cdot\cos\pi t - \frac{1}{\pi}\sin\pi x\cdot\sin\pi t$$

$$U(x,t) = \sin\pi x\left[\cos\pi t - \frac{\sin\pi t}{\pi}\right] \text{ Answer}$$

** **

Question 4:- $U_{tt}(x,t) = a^2 U_{xx}(x,t)$, $(t>0,\ x>0)$ — (1)

$$U(x,0) = U_t(x,0) = 0 \text{ — (2)}$$

$$U(0,t) = f(t) \ , \ \lim_{x\to\infty} U(x,t) = 0 \text{ — (3)}$$

Solution:- Taking Laplace Transform of eqn (1)

$$\mathcal{L}\{U_{tt}(x,t)\} = a^2\mathcal{L}\{U_{xx}(x,t)\}$$

$$s^2U(x,s) - sU(x,0) - U_t(x,0) = a^2U_{xx}(x,s)$$

$$s^2U(x,s) - 0 + 0 = a^2U_{xx}(x,s)$$

$$\Rightarrow U_{xx}(x,s) - \frac{s^2}{a^2}U(x,s) = 0$$

which is a second order Homogeneous D-E whose Solution is given by

$$\left[D^2 - \frac{s^2}{a^2}\right]U(x,s) = 0$$

It's characteristic equation is

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$$m^2 - \frac{s^2}{a^2} = 0 \Rightarrow m = \pm s/a$$

Thus $U(x,s) = c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x}$ ——— (4)

Now from (3)

$U(0,t) = f(t)$ and $\lim_{x\to\infty} U(x,s) = 0$ ——— (5)

$\mathcal{L}\{U(0,t)\} = \mathcal{L}\{f(t)\}$

$U(0,s) = F(s)$ ——— (6)

From (4) and (5), we have

$\lim_{x\to\infty} U(x,s) = c_1 e^{\infty} + c_2 e^{-\infty}$

$0 = c_1(\infty) + 0$

$\Rightarrow \boxed{c_1 = 0}$

Hence $U(x,s) = c_2 e^{-\frac{s}{a}x}$

Putting $x = 0$

$U(0,s) = c_2 e^0$

$F(s) = c_2$ using (6)

Hence $U(x,s) = F(s) e^{-\frac{x}{a}s}$

Now Taking Laplace inverse on both side

$$\mathcal{L}^{-1}\{U(x,s)\} = \mathcal{L}^{-1}\{e^{-\frac{x}{a}s} F(s)\}$$

$U(x,t) = H(t-\frac{x}{a}) f(t-\frac{x}{a})$ → By 2nd Shifting Th

where

$$H(t-\tfrac{x}{a}) f(t-\tfrac{x}{a}) = \begin{cases} f(t-\frac{x}{a}) & ; t > x/a \\ 0 & ; t < x/a \end{cases}$$

Question 5 :- Find the formal solution of the problem $K U_{xx} = U_t$, $t > 0$, $0 < x < \infty$

$U(0,t) = f(t)$, $U(x,0) = 0$

$\lim_{x\to\infty} U(x,t) = 0$

Solution :- Given that $K U_{xx} = U_t$

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Taking Laplace Transform

$$K\mathcal{L}\{U_{xx}\} = \mathcal{L}\{U_t\}$$

$$K[U_{xx}(x,s)] = sU(x,s) - U(x,0)$$

$$K U_{xx}(x,s) = sU(x,s) - 0$$

$$U_{xx}(x,s) - \frac{s}{K} U(x,s) = 0$$

Which is Homogeneous Second order D-E whose solution is given by

$$\left(D^2 - \frac{s}{K}\right) U(x,s) = 0$$

The auxiliary equation is

$$m^2 - \frac{s}{K} = 0 \Rightarrow m = \pm\sqrt{\frac{s}{K}}$$

Thus

$$U(x,s) = C_1 e^{\sqrt{\frac{s}{K}}x} + C_2 e^{-\sqrt{\frac{s}{K}}x} \quad \text{---} (1)$$

Now $U(0,t) = f(t)$

$$\mathcal{L}\{U(0,t)\} = \mathcal{L}\{f(t)\}$$

$$U(0,s) = F(s)$$

Put $x = 0$ in (1)

$$U(0,s) = C_1 + C_2$$

$$F(s) = C_1 + C_2$$

$$F(s) = C_2$$

$\lim_{x\to\infty} U(x,t) = 0$

$\lim_{x\to\infty} U(x,s) = 0$

$\lim_{x\to\infty} U(x,s) = C_1 e^{\infty} + C_2 e^{-\infty}$

$0 = C_1(\infty) + 0$

$\Rightarrow \boxed{C_1 = 0}$

These for $U(x,s) = F(s) e^{-\sqrt{\frac{s}{K}}x}$

Taking Laplace Inverse on both side

$$\mathcal{L}^{-1}\{U(x,s)\} = \mathcal{L}^{-1}\{e^{-x\sqrt{\frac{1}{K}}\sqrt{s}} \cdot F(s)\}$$

$$U(x,t) = \mathcal{L}^{-1}\left\{e^{-\frac{x}{\sqrt{K}}\sqrt{s}} \cdot F(s)\right\}$$

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FOURIER TRANSFORM AND Its APPLICATION

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Definition :- If $f(x)$ is integrable for $-\infty < x < \infty$ we can associate with it another function $F(K)$. $K \in [$ by the relation $\mathcal{F}\{f(x)\} = F(K) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iKx} f(x)\,dx$

The function $F(K)$ is called Fourier Transform of $f(x)$ and $\mathcal{F}$ is called Fourier Transform Operator.

Fouriex Inverse Transform

The function $f(x)$ is called inverse Fourier Transform of $F(K)$

$$f(x) = \mathcal{F}^{-1}\{F(K)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iKx} F(K)\,dK$$

Existance [Foriex & Inverse Fourier Transform]

If the function $f(x)$ or $F(K)$ is continous or Piecewise continous over $]-\infty, \infty[$ and bounded

i-e $|f(x)| \leq M$ as $x \to \infty$

Then Fouriex Transform and inverse Fourier Transfo. exist.

Absolutely Integrable function

If the function $f(x)$ is absolutly integrable ($\int_{-\infty}^{\infty} |f(x)|\,dx$ exists) Then Its Foriex Transform exists and vice versa.

$|f(x)| \to 0$ as $x \to \infty$

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(230)

PROPERTIES OF FOURIER TRANSFORMS

1) Linearity Property

The Fourier transform operator F is linear. i-e

If "a" and "b" are constants while $g(x)$ and $h(x)$ are function of "x". Then

$$F\{a g(x) + b h(x)\} = a F\{g(x)\} + b F\{h(x)\}$$

Proof:- Let $f(x) = a g(x) + b h(x)$

Then

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iKx} f(x) \cdot dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iKx} [a g(x) + b h(x)] \cdot dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[a e^{iKx} g(x) + b e^{iKx} h(x) \right] dx$$

$$= \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iKx} g(x) + \frac{b}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iKx} h(x)\, dx$$

$$= a F\{g(x)\} + b F\{h(x)\}$$

2) Conjugation property

If $f(x)$ is real, Then $F(-K) = \overline{F(K)}$

Proof :- $F(K) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iKx} f(x) \cdot dx$

Taking Conjugate

$$\overline{F(K)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iKx} f(x) \cdot dx$$

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$$\overline{F(K)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{e^{2iKx}} \cdot \overline{f(x)} \cdot dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2iKx} f(x) \cdot dx \quad (\because f(x) \text{ is real } \overline{f(x)} = f(x))$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{2i(-K)x} f(x) \cdot dx .$$

$\overline{F(K)} = F(-K)$ Hence prove

⟺

A/2014

Real and Complex Value of Fourier Transform

(a) If $f(x)$ is real and even Then $F(K)$ is real

Proof:- Let $f(x)$ is Real and even

$$F(K) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{2iKx} f(x) \cdot dx$$

Since $f(-x) = f(x)$ Becaus $f(x)$ is even

There for $$F(K) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{2iKx} f(-x) \cdot dx$$

put $-x = x' \Rightarrow dx = -dx'$

when $x \to \infty$ Then $x' \to -\infty$

when $x \to -\infty$ Then $x' \to +\infty$

$$F(K) = \frac{1}{\sqrt{2\pi}} \int_{+\infty}^{-\infty} e^{-2iKx'} f(x') (-dx')$$

$$= \frac{-1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} e^{-2iKx'} f(x')\, dx'$$

$$= \frac{+1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{2i(-K)x'} f(x')\, dx'$$

$$F(K) = \overline{F(K)}$$

which Shows that $F(K)$ is Real.

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A/2014, A/2008

b) If $f(x)$ is Real and odd then $F(K)$ is Pure imaginary.

Proof:- If $f(x)$ is Real and odd then

$$f(-x) = -f(x)$$

or $$-f(-x) = f(x)$$

As we know that

$$F(K) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iKx} f(x)\, dx$$

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$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iKx} [-f(-x)]\, dx$$

$$F(K) = \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iKx} f(-x)\, dx$$

put $-x = x' \Rightarrow dx = -dx'$

when $x \to \infty$ Then $x' \to -\infty$

$x \to -\infty$ Then $x' \to \infty$

$$F(K) = \frac{-1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} e^{-iKx'} f(x')(-dx')$$

$$= \frac{+1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} e^{-iKx'} f(x')\, dx'$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(-K)x'} f(x')\, dx'$$

$$F(K) = -F(-K)$$

Since from Conjugation property $\overline{F(K)} = F(-K)$

$$\overline{F(K)} = -F(K)$$

or $$\overline{F(K)} = -F(K)$$

which shows that $F(K)$ is Pure imaginary

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(c.) If $f(x)$ is complex then

$$\mathcal{F}\{\overline{f(-x)}\} = \overline{F(K)}$$

Proof :-

$$\mathcal{F}\{f(-x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iKx}\{\overline{f(-x)}\}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iKx}\,\bar{f}(-x)\cdot dx$$

Put $-x = x' \Rightarrow dx = -dx'$

And when $x \to \infty$ Then $x' \to -\infty$

$x \to -\infty$ Then $x' \to \infty$

$$\mathcal{F}\{f(-x)\} = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} e^{-iKx'}\,\bar{f}(x')(-dx')$$

$$= \frac{\pm 1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iKx'}\,\bar{f}(x')\cdot dx'$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{e^{iKx} f(x')\cdot dx'}$$

$$= \text{Complex conjugate of } \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iKx'} f(x')dx$$

$$= \text{Complex conjugate of } F(K)$$

$$\mathcal{F}\{\overline{f(-x)}\} = \overline{F(K)}$$

(4) Attenuation property :- $\mathcal{F}\{e^{ax}f(x)\} = F(K-ia$

Proof :-

$$\mathcal{F}(e^{ax}f(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iKx}\left[e^{ax}f(x)\right]\cdot dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{(iK+a)x} f(x)\cdot dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(K-ia)x} f(x)\cdot dx = F(K-ia$$

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1/2011, S/2010, A/2010, A/2007

5) Scaling Property :- If c is a non-zero constant Then $\mathcal{F}\{f(cx)\} = \frac{1}{|c|}\mathcal{F}\left(\frac{K}{c}\right)$

Proof:- Let $c > 0$ Then

$$\mathcal{F}\{f(cx)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iKx} f(x)\,dx$$

Put $cx = x' \Rightarrow x = \frac{x'}{c}$

$$dx = \frac{dx'}{c}$$

and $x \to \pm\infty$ Then $x' \to \pm\infty$

$$\mathcal{F}\{f(cx)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iK\left(\frac{x'}{c}\right)} f(x')\frac{dx'}{c}$$

$$= \frac{1}{c}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\left(\frac{K}{c}\right)x'} f(x')\cdot dx'$$

$$= \frac{1}{c}\mathcal{F}\left(\frac{K}{c}\right), \quad c > 0$$

If $c < 0$ Then $c = -z$, $z > 0$

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iKx} f(-zx)\,dx$$

put $-zx = x'$

$$x = -\frac{1}{z}x' \Rightarrow dx = -\frac{1}{z}dx'$$

when $x \to \infty$ Then $x' \to -\infty$
$x \to -\infty$ Then $x' \to \infty$

$$= \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} e^{iK\left(\frac{x'}{z}\right)} f(x')\left(\frac{-dx'}{z}\right)$$

$$= \frac{-1}{z}\frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} e^{i\left(\frac{K}{-z}\right)x'} f(x')\,dx'$$

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$$= \frac{\pm 1}{z\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{2\left(\frac{K}{-z}\right)x'} f(x')\,dx'$$

$$= \frac{1}{z} F\left(\frac{K}{-z}\right)$$

since $c = -z$, $z > 0$

$$= \frac{-1}{c} F\left(\frac{K}{c}\right)$$

Combining the two results we have

$$F\{f(x)\} = \frac{1}{|c|} F\left(\frac{K}{c}\right)$$

(6) Shifting Property

A/2010 (i) $F\{f(x-a)\} = e^{2Ka} F(K)$

Proof:- $F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iKx} f(x-a)\,dx$

put $x - a = x' \Rightarrow dx = dx'$

when $x \to \infty$ Then $x' \to \infty$

$x \to -\infty$ Then $x' \to -\infty$

$$F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iK(x'+a)} f(x')\,dx'$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{2Kx'} \cdot e^{2Ka} f(x')\,dx'$$

$$= \frac{e^{2Ka}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{2Kx'} f(x)\,dx'$$

$$= e^{2Ka} F(K)$$

6(b)

S/2010 (ii) $F\{e^{iax} f(x)\} = F(K+a)$

Proof:- $F\{e^{iax} f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iKx} \cdot e^{iax} f(x)\,dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(K+a)x} f(x)\,dx$$

$$= F(K+a)$$

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7) Modulation Property

(i) $\mathcal{F}\{\cos\alpha x\, f(x)\} = \frac{1}{2}\{F(K+\alpha) + F(K-\alpha)\}$

Proof :-

$$\mathcal{F}\{\cos\alpha x\, f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iKx}\cos\alpha x\, f(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iKx}\left[\frac{e^{i\alpha x} + e^{-i\alpha x}}{2}\right] f(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iKx}\cdot\frac{e^{i\alpha x}}{2} f(x)\,dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iKx}\cdot\frac{e^{-i\alpha x}}{2} f(x)\,dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(K+\alpha)x} f(x)\,dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(K-\alpha)x} f(x)\,dx\right]$$

$$= \frac{1}{2}\left[F(K+\alpha) + F(K-\alpha)\right]$$

(ii) $\mathcal{F}\{\sin\alpha x\, f(x)\} = \frac{1}{2i}\left[F(K+\alpha) - F(K-\alpha)\right]$

Proof :- $\mathcal{F}\{\sin\alpha x\, f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iKx}\sin\alpha x\, f(x)\,dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iKx}\left[\frac{e^{i\alpha x} - e^{-i\alpha x}}{2i}\right] f(x)\,dx$$

$$= \frac{1}{2i}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iKx}\left[e^{i\alpha x} - e^{-i\alpha x}\right] f(x)\,dx$$

$$= \frac{1}{2i}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iKx}\cdot e^{i\alpha x} f(x).\,dx - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iKx}\cdot e^{-i\alpha x} f(x)\,dx\right]$$

$$= \frac{1}{2i}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(K+\alpha)x} f(x).\,dx - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(K-\alpha)x} f(x)\,dx\right]$$

$$= \frac{1}{2i}\left[F(K+\alpha) - F(K-\alpha)\right]$$

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A/2016, A/2008, A/2007

Example#1 :- Fourier transform of Gaussian funct.

$$g(x) = Ne^{-\alpha x^2}$$

where N and α are constants and $\alpha > 0$

Solution:-

$$F\{g(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{2\pi i x} \left[Ne^{-\alpha x^2}\right] dx$$

$$= \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(-\alpha x^2 + 2\pi i x)} \cdot dx$$

$$= \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha\left[x^2 - \frac{2\pi i}{\alpha} x\right]} \cdot dx$$

$$= \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha\left[x^2 - 2(x)\left(\frac{\pi i}{\alpha}\right) + \left(\frac{\pi i}{\alpha}\right)^2 - \left(\frac{\pi i}{\alpha}\right)^2\right]} \cdot dx$$

$$= \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha\left[\left(x - \frac{\pi i}{\alpha}\right)^2 - \frac{\pi^2 i^2}{\alpha^2}\right]} \cdot dx$$

$$= \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha\left(x - \frac{\pi i}{\alpha}\right)^2 - \frac{\pi^2}{\alpha}} \cdot dx$$

$$= \frac{N}{\sqrt{2\pi}} \cdot e^{\frac{-\pi^2}{\alpha}} \int_{-\infty}^{\infty} e^{-\alpha\left(x - \frac{\pi i}{\alpha}\right)^2} \cdot dx \quad \text{---(1)}$$

put $\alpha\left(x - \frac{\pi i}{\alpha}\right)^2 = t^2$

$$t = \sqrt{\alpha} \cdot \left(x - \frac{\pi i}{\alpha}\right)$$

$$dt = \sqrt{\alpha} \cdot dx$$

$$\frac{dt}{\sqrt{\alpha}} = dx$$

putting these in (1)

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$$\mathcal{F}\{g(x)\} = \frac{N}{\sqrt{2\pi}} e^{-K^2/4\alpha} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{\sqrt{\alpha}}$$

$$= \frac{N}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{\alpha}} e^{-K^2/4\alpha} \int_{-\infty}^{\infty} e^{-t^2} \cdot dt$$

$$= \frac{N}{\sqrt{2\alpha\pi}} e^{\frac{-K^2}{4\alpha}} \sqrt{\pi}$$

$$\mathcal{F}\{g(x)\} = \frac{N}{\sqrt{2\alpha}} e^{\frac{-K^2}{4\alpha}}$$

Gaussian integral

$$\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$$

A/2009

Fouriex Transforms of Dexivatives

Prove that $\mathcal{F}\{f^n(x)\} = (-iK)^n \mathcal{F}(K)$

Proof :-

$$\mathcal{F}\{f'(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iKx} f'(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[e^{iKx} f(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x) e^{iKx} (iK)\, dx \right]$$

$f(x)$ is sppосed to tends to zero as $x \to \pm\infty$

$$= \frac{1}{\sqrt{2\pi}} \left[0 - iK \int_{-\infty}^{\infty} e^{iKx} f(x) \cdot dx \right]$$

$$= (-iK) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iKx} f(x) \cdot dx$$

$$\mathcal{F}\{f'(x)\} = (-iK)\, \mathcal{F}(K) \quad \text{——} (1)$$

Now $\mathcal{F}\{f''(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iKx} f''(x) \cdot dx$

$$= \frac{1}{\sqrt{2\pi}} \left[e^{iKx} f'(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x) e^{iKx} (iK)\, dx \right]$$

Supposed $f'(x) \to 0$ as $x \to \mp\infty$

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$$F\{f''(x)\} = \frac{1}{\sqrt{2\pi}}\left[0 - (iK)\int_{-\infty}^{\infty} e^{iKx} f'(x)\cdot dx\right.$$

$$= (-iK)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iKx} f'(x)\cdot dx$$

$$= (-iK)\,F\{f'(x)\}$$

$$= (-iK)(-iK)\,F(K) \qquad [\because \text{From } (1)]$$

$$F\{f''(x)\} = (-iK)^2 F(K) \quad \text{---} (2)$$

$\therefore$ Now $F\{f'''(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iKx} f'''(x)\cdot dx$

$$= \frac{1}{\sqrt{2\pi}}\left[e^{iKx} f''(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f''(x)\, e^{iKx}(iK)\, d\right.$$

Supposed $f''(x) \to 0$ as $x \to \pm\infty$

$$= \frac{1}{\sqrt{2\pi}}\left[0 - iK\int_{-\infty}^{\infty} e^{iKx} f''(x)\cdot dx\right]$$

$$= (-iK)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iKx} f''(x)\cdot dx$$

$$= (-iK)\,F\{f''(x)\}$$

$$= (-iK)(-iK)^2 F(K) \quad \because \text{using } (2)$$

$$F\{f''(x)\} = (-iK)^3 F(K)$$

Similarly Continuing this process we get

$$F\{f^n(x)\} = (-iK)^n F(K)$$

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Fourier Transform of the function of form $x^n f(x)$

$$\mathcal{F}\{x^n f(x)\} = (-i)^n \frac{d^{(n)}}{dx^{(n)}} F(K) = (-i)^n F^{(n)}(K)$$

Solution :-

$$\mathcal{F}(K) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{2\pi K x} f(x) \cdot dx$$

Differentiate . w . r . t "K"

$$F'(K) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) \cdot e^{2\pi K x} (2ix) \cdot dx$$

$$= i \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{2\pi i K x} \, x f(x) . dx$$

$$F'(K) = i \, F(x f(x))$$

$$\frac{1}{i} F'(K) = F(x f(x))$$

$$(-i) F'(K) = F(x f(x))$$

OR $$F\{x f(x)\} = (-i) F'(K)$$

$$F\{x^2 f(x)\} = (-i)^2 F''(K)$$

$$F\{x^3 f(x)\} = (-i)^3 F'''(K)$$

$$F\{x^n f(x)\} = (-i)^n F^n(K)$$

Hence prove

Fourier Transform of Integrals

Suppose that $f(x)$ is piecewise continous on $(-\infty, \infty)$ and absolutely integrable. Also let $F(0) = 0$ with $F(K) = F\{f(x)\}$ Then

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$$= F\left\{\int_{-\infty}^{x} f(x')\,dx' = \frac{1}{2K} F(K)\right.$$

Proof:-

Let $\phi(x) = \int_{-\infty}^{x} f(x')\,dx'$

$\Rightarrow \phi'(x) = f(x)$

Firstly, we will check if Fourier transform exists

$\lim_{x\to\infty} \phi(x) = \int_{-\infty}^{\infty} f(x')\,dx' = \sqrt{2\pi}\,F(0) = 0$

$\therefore F(K) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2\pi i K x} f(x)\,dx$

$\therefore F(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{0} f(x)\,dx$

$\therefore \sqrt{2\pi}\,F(0) = \int_{-\infty}^{\infty} f(x)\,dx$

$\lim_{x\to\infty} \phi(x) = 0$

$\therefore$ As $\mathcal{F}\{\phi'(x)\} = (-2iK)\,\mathcal{F}\{\phi(x)\}$

$\therefore \mathcal{F}\{f(x)\} = F(K) = (-2iK)\,\Psi(K)$ ....

where $[\because \mathcal{F}\{\phi(x)\} = \Psi(K)]$

$\Rightarrow \Psi(K) = \frac{F(K)}{-2iK}$

$\Psi(K) = \frac{iF(K)}{K}$

A/2010

CONVOLUTION :- If $f(x)$ and $g(x)$ are function of x defined over the interval $(-\infty, \infty)$, Then their Convolution denoted by $f*g$ and defined

$$f*g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tau)\,g(t-\tau)\,d\tau = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t-\tau)\,g(\tau)\,d\tau$$

Some properties can easily verified that

(1) $f*g = g*f$ (2) $f*(g+h) = f*g + f*h$ (3) $f*(g*h) =$

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A/2008, S/2009

CONVOLUTION THEOREM

A/2014

Statment :- If $F(K)$ and $G(K)$ are fourier transform of $f(x)$ and $g(x)$ respectively then

$$\mathcal{F}\{f * g\} = F(K)\,G(K)$$

OR

$$f * g = \mathcal{F}^{-1}\{F(K)\cdot G(K)\}$$

Proof :- Suppose that $F(K)$ and $G(K)$ be the fourier transform of $f(x)$ and $g(x)$ respectively then by definition

$$\mathcal{F}^{-1}\{F(K)\cdot G(K)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iKx}\,F(K)G(K)\,dK$$

$$\mathcal{F}^{-1}\{F(K)\cdot G(K)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iKx}\,F(K)\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iKx'}\,g(x')\,dx'\right\}dK$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iK(x-x')}\,F(K)\,dK\right\}g(x')\,dx'$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left\{\; f(x-x') \;\right\}g(x')\,dx'$$

$$\mathcal{F}^{-1}\{F(K)G(K)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-x')\cdot g(x')\cdot dx'$$

$$\mathcal{F}^{-1}\{F(K)G(K)\} = f * g$$

which is equivalent to

$$F(K)G(K) = \mathcal{F}\{f * g\}$$

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PARSEVAL'S FIRST THEOREM

Statment :- $\int_{-\infty}^{\infty} F(K)\,\overline{F(K)}\,dK = \int_{-\infty}^{\infty} f(x)\cdot\overline{f(x)}\,dx$

or $\int_{-\infty}^{\infty} |F(K)|^2\,dK = \int_{-\infty}^{\infty} |f(x)|^2\,dx$

Proof :- By Convolution Theorem

$$F^{-1}\{F(K)\,G(K)\} = f(x) * g(x)$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-2\pi K u} F(K)\,G(K)\,dK = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,g(x-u)\,du$$

Put $x=0$ on both side

$$\int_{-\infty}^{\infty} e^{0} F(K)\,G(K)\,dK = \int_{-\infty}^{\infty} f(u)\,g(0-u)\,du$$

$$\int_{-\infty}^{\infty} F(K)\,G(K)\,dK = \int_{-\infty}^{\infty} f(u)\,g(-u)\,du$$

replace "u" by "x"

$$= \int_{-\infty}^{\infty} f(x)\,g(-x)\,dx$$

Let $g(-x) = \overline{f(x)}$ i.e $g(x) = \overline{f(-x)}$

Therefor $F\{g(x)\} = F\{\overline{f(-x)}\}$

$$G(x) = \overline{F(K)}$$

Hence $\int_{-\infty}^{\infty} F(K)\cdot\overline{F(K)}\,dK = \int_{-\infty}^{\infty} f(x)\,\overline{f(x)}\,dx$

$$\int_{-\infty}^{\infty} |F(K)|^2\,dK = \int_{-\infty}^{\infty} |f(x)|^2\,dx$$

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W/2011, S/2010, S/2009, A/2009

PARSEVAL'S SECOND THEOREM

Statment :-

$$\int_{-\infty}^{\infty} F(K)\,G(K)\,dK = \int_{-\infty}^{\infty} f(u)\,g(-u)\,du = \int_{-\infty}^{\infty} f(u)\,g(-u)\,du$$

Proof :- By Convolution Theorem

$$\mathcal{F}^{-1}\{F(K)\,G(K)\} = f(x) * g(x)$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iKx} F(K)\,G(K)\,dK = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,g(x-u)\,du$$

putting $x = 0$ on both sides

$$\int_{-\infty}^{\infty} e^{0} F(K)\,G(K)\,dK = \int_{-\infty}^{\infty} f(u)\,g(0-u)\,du$$

$$\int_{-\infty}^{\infty} F(K)\,G(K)\,dK = \int_{-\infty}^{\infty} f(u)\,g(-u)\,du$$

Now "u" replace by "x"

$$= \int_{-\infty}^{\infty} f(x)\,g(-x)\,dx$$

Hence

$$\int_{-\infty}^{\infty} F(K)\,G(K)\,dK = \int_{-\infty}^{\infty} f(u)\,g(-u)\,du = \int_{-\infty}^{\infty} f(x)\,g(-x)\,dx$$

*** ***

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Example :- Find the Fourier transform of

$$g(x) = \frac{a}{x^2+a^2}, \quad a > 0$$

Solution:- $\mathcal{F}\{g(z)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikz} \frac{a}{z^2+a^2} dz$

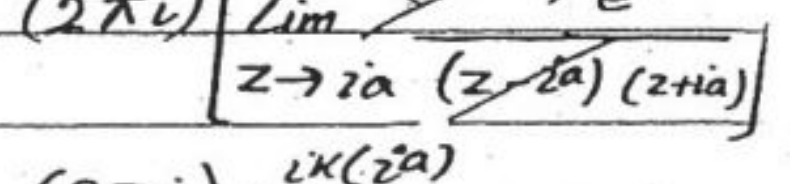
$$= \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikz} \frac{1}{z^2+a^2} dz$$

$$= \frac{a}{\sqrt{2\pi}} \oint_C e^{ikz} \frac{1}{z^2+a^2} dz$$

where C is a closed contour containing x-axis and a semi-circle (of infinite radius) in upper or lower half.

$e^{ikz} = e^{ik(x+iy)} = e^{ikx-ky} \to 0$ As $y \to \infty$

$e^{ikx-ky} \to 0$ as $y \to -\infty$



we want to choose a contour C, such that the integral $\oint_C g(z)dz = 0$

$$G(K) = \frac{a}{\sqrt{2\pi}} \oint_C \frac{e^{ikz}}{z^2+a^2} \cdot dz$$

$$= \frac{a}{\sqrt{2\pi}} \times 2\pi i \times S$$

where S is sum of Residues

(i) when $K > 0$, $z = ia$

(ii) when $K < 0$, $z = -ia$

Case I $(K>0) \Rightarrow G(K) = \frac{a}{\sqrt{2\pi}} (2\pi i) \left[\lim_{z \to ia} \frac{(z-ia)\, e^{ikz}}{(z-ia)(z+ia)} \right]$

$$= \frac{a}{\sqrt{2\pi}} (2\pi i) \frac{e^{ik(ia)}}{(ia+ia)}$$

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$$G(K) = \frac{a}{\sqrt{2\pi}} 2\pi i \frac{e^{-Ka}}{2ia}$$

$$G(K) = \frac{\pi}{\sqrt{2\pi}} e^{-Ka}, \quad K > 0$$

$$G(K) = \sqrt{\frac{\pi}{2}} e^{-Ka}$$

Case II (K < 0)

$$G(K) = \frac{a}{\sqrt{2\pi}} (2\pi i) \lim_{z \to -ia} \left[\frac{(z+ia) e^{ikz}}{(z-ai)(z+ia)} \right]$$

$$= \frac{a}{\sqrt{2\pi}} (2\pi i) \cdot \lim_{z \to -ia} \left[\frac{e^{ikz}}{(z-ia)} \right]$$

$$= \frac{a}{\sqrt{2\pi}} (2\pi i) \frac{e^{ik(-ia)}}{-ia-ia}$$

$$= \frac{a}{\sqrt{2\pi}} (2\pi i) \frac{e^{-i^2 ka}}{-2ia}$$

$$= -\frac{\pi}{\sqrt{2\pi}} e^{ka}$$

$$G(K) = -\sqrt{\frac{\pi}{2}} e^{ka}$$

Combining the two results we have

$$G(K) = \sqrt{\frac{\pi}{2}} \left(e^{-Ka} - e^{Ka} \right)$$

1/2012

Question :- calculate the fourier transform of function $f(x) = \frac{1}{x^2+4}$

Solution :- The function $f(x)$ and its derivative are continous over the interval $(-\infty, \infty)$ and their integral $\int_{-\infty}^{\infty} f(x)\,dx$ is absolutely integerable. Therefor Fourier transform of given function exist.

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$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iKx} \frac{1}{x^2+4}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \oint_{c} \frac{e^{iKz}}{z^2+2^2}$$

where "c" is a closed contour containing x-axis a semi circle of infinite radius (in the upper or lower half).

∴ Therefore $e^{iKz} = e^{2iK(x+iy)} = e^{2iKx - Ky} \longrightarrow 0$ as $y \to +\infty$

we want to choose a Contour "c" such that the integral $\int_c f(z)\, dz$ is zero

$$\mathcal{F}(K) = \frac{1}{\sqrt{2\pi}} \oint \frac{e^{iKz}}{z^2+2^2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} 2\pi i \times S$$

where S is sum of residue

(i) $K > 0$ $\quad z = 2i$

(ii) $K < 0$ $\quad z = -2i$

Case I, when ($K > 0$) $= F(K) = \frac{1}{\sqrt{2\pi}} (2\pi i) \left[\lim_{z \to 2i} \frac{(z-2i)\, e^{iKz}}{(z+2i)(z-2i)} \right]$

$$= \frac{1}{\sqrt{2\pi}} (2\pi i) \frac{e^{-2K}}{2i+2i}$$

$$= \frac{1}{\sqrt{2\pi}} (2\pi i) \frac{e^{-2K}}{4i}$$

$$= \frac{1}{2} \sqrt{\frac{\pi}{2}}\, e^{-2K} \qquad (i)$$

Case II, when ($K < 0$)

$$F(K) = \frac{1}{\sqrt{2\pi}} (2\pi i) \left[\lim_{z \to -2i} \frac{(z+2i)\, e^{2iKz}}{(z+2i)(z-2i)} \right]$$

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$$\mathcal{F}\{K\} = \frac{1}{\sqrt{2\pi}}(2\pi i)\ \frac{e^{2K}}{-2i-2i}$$

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$$= \frac{1}{\sqrt{2\pi}}(2\pi i)\ \frac{e^{2K}}{-4i}$$

$$= -\frac{1}{2}\sqrt{\frac{\pi}{2}}\, e^{2K} \qquad \text{(ii)}$$

By Combining (i) & (ii)

$$F(K) = \frac{1}{2}\sqrt{\frac{\pi}{2}}\left[e^{-2K} - e^{2K}\right]$$

$$= \sqrt{\frac{\pi}{2}}\left[-\frac{(e^{2K} - e^{-2K})}{2i}\right]$$

$$= \frac{\sqrt{\pi}}{2}\left[-\sinh 2K\right]$$

$$= -\sqrt{\frac{\pi}{2}}\sinh 2K$$

FOURIER INTEGRAL THEOREM

tatment:- If $f(x)$ is real function defined ver $]-\infty, \infty[$ and the integral $\int_{-\infty}^{\infty} f(x)\,dx$ is bsolutly convergent then

$$f(x) = \frac{1}{\pi}\int_0^{\infty} dK \int_{-\infty}^{\infty} \cos K(x-x')\, f(x')\, dx'$$

Proof:- Since the integral $\int_{-\infty}^{\infty} f(x)dx$ is absolutly convergent, its fourier and verse fourier transform both exist. Therefor

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iKx}\, \mathcal{F}(K)\, dK$$

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$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{-iKx} F(K)\, dK + \int_{0}^{\infty} e^{-iKx} F(K)\, dK \right] \quad \text{--- (1}$$

Take $I_1 = \int_{-\infty}^{0} e^{-iKx} F(K)\, dK$

Let $-K = K'$

$dK = -dK'$

$K \longrightarrow -\infty$, $K' \longrightarrow \infty$

$K \longrightarrow 0$, $K' \longrightarrow 0$

$$I_1 = -\int_{\infty}^{0} e^{iK'x} F(-K')\, dK'$$

$\therefore$ If $F(K)$ is real then $F(-K) = \overline{F(K)}$

$$I_1 = \int_{0}^{\infty} e^{iK'x} \overline{F(K')}\, dK' = \int_{0}^{\infty} e^{iKx} \overline{F(K)}\, dK$$

put in (1)

$$f(x) = \frac{1}{\sqrt{2\pi}} \left[\int_{0}^{\infty} e^{iKx} \overline{F(K)}\, dK + \int_{0}^{\infty} e^{-iKx} F(K)\, dK \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{0}^{\infty} \left\{ e^{iKx} \overline{F(K)} + e^{-iKx} F(K) \right\} dK \right] \quad \text{--- (2)}$$

As

$$e^{iKx} \overline{F(K)} = \frac{e^{iKx}}{\sqrt{2\pi}} \overline{\int_{-\infty}^{\infty} e^{iKx'} f(x')\, dx'}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iKx} \cdot e^{-iKx'} f(x')\, dx'$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iK(x-x')} f(x')\, dx'$$

Also

$$e^{-iKx} F(K) = \frac{e^{-iKx}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iKx'} f(x')\, dx'$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iKx} \cdot e^{iKx'} f(x')\, dx'$$

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$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iK(x-x')} f(x')\,dx'$$

put in (2)

$$f(x) = \frac{1}{\sqrt{2\pi}} \left[\int_0^{\infty} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iK(x-x')} f(x')\,dx' + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iK(x-x')} f(x')\,dx' \right] dK \right]$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x') \left(e^{iK(x-x')} + e^{-iK(x-x')} \right) dx'\,dK \right]$$

$$f(x) = \frac{1}{2\pi} \int_0^{\infty} \left\{ \int_{-\infty}^{\infty} 2 \cos K(x-x')\, f(x')\, dx'\, dK \right.$$

$$= \frac{1}{\pi} \int_0^{\infty} dK \int_{-\infty}^{\infty} \cos K(x-x')\, f(x')\, dx'$$

* * * *

FOURIER Sin & Cosine TRANSFORMS

$$F_S(K) = \mathcal{F}_S\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin Kx\, dx$$

$$F_C(K) = \mathcal{F}_C\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos Kx\, dx$$

* * * *

INVERSE FOURTER

Sin & Cosine TRANSFORM

$$f(x) = \mathcal{F}_S^{-1}\{F(K)\} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F(K) \sin Kx\, dK$$

$$f(x) = \mathcal{F}_C^{-1}\{F(K)\} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F(K) \cos Kx\, dK$$

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S/2010, A/2009

Question:- Express $F_c\{f''(x); K\}$ in term of fourier Sine or Cosine transform.

Solution:- Assume that $f(x)$ is real and $|f'(x)| \to 0$ as $x \to \infty$ Then

$$F_c\{f''(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f''(x) \cos Kx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\cos Kx f(x) \Big|_0^\infty - \int_0^\infty f(x)(-K \sin Kx) dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\{0 - (1) f'(0)\} + K \int_0^\infty f'(x) \sin(Kx) dx \right.$$

$$= -\sqrt{\frac{2}{\pi}} f'(0) + K \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(Kx) \, dx$$

$$= -\sqrt{\frac{2}{\pi}} f'(0) + K \sqrt{\frac{2}{\pi}} \left[\sin Kx f(x) \Big|_0^\infty - \int_0^\infty f(x) K \cos Kx \right.$$

$$= -\sqrt{\frac{2}{\pi}} f'(0) + K \sqrt{\frac{2}{\pi}} \left[0 - K \int_0^\infty f(x) \cos Kx \, dx \right.$$

$$= -\sqrt{\frac{2}{\pi}} f'(0) + 0 - K^2 \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos Kx \, dx$$

$$F_c\{f''(x)\} = -\sqrt{\frac{2}{\pi}} f'(0) - K^2 F_c(K)$$

N2011, S/2009

Question:- Express $F_s\{f''(x); K\}$ in term of Fourier Sine or Cosine Transform.

Solution:- Assume that $f(x)$ is real and $|f'(x)| \to 0$ as $x \to \infty$ Then

$$F_s\{f''(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f''(x) \sin Kx \, dx$$

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$$= \sqrt{\frac{2}{\pi}} \left[\sin Kx\, f'(x) \Big|_0^\infty - \int_0^\infty f'(x)(K \cos Kx)\, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - K \int_0^\infty f'(x) \cos Kx\, dx \right.$$

$$= -K \sqrt{\frac{2}{\pi}} \left[\cos Kx\, f(x) \Big|_0^\infty - \int_0^\infty f(x)(-K \sin Kx)\, dx \right.$$

$$= -K \sqrt{\frac{2}{\pi}} \left[\{0 - (1) f(0)\} + K \int_0^\infty f(x) \sin Kx\, dx \right.$$

$$= K \sqrt{\frac{2}{\pi}} f(0) - K^2 \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin Kx\, dx$$

$$F_s\{f''(x)\} = K \sqrt{\frac{2}{\pi}} f(0) - K^2 F_s(K)$$

Example :- Show that

(a) $$F_s\{x e^{-ax}\} = \sqrt{\frac{2}{\pi}} \frac{2aK}{(a^2+K^2)^2}$$

Solution:-

Take L.H.S $= F_s\{x e^{-ax}\}$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty x e^{-ax} \sin Kx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty x (e^{-ax} \sin Kx)\, dx$$

using the formula

$$\int e^{ax} \sin bx = \frac{e^{ax}}{a^2+b^2} [a \sin bx - b \cos bx]$$

$$\text{L.H.S} = \sqrt{\frac{2}{\pi}} \left[\left\{ x \cdot \frac{e^{-ax}}{a^2+K^2} (-a \sin Kx - K \cos Kx) \Big|_0^\infty - \right. \right.$$

$$\int \frac{e^{-ax}}{a^2+K^2} (-a \sin Kx - K \cos Kx) \cdot 1 \cdot dx \Big]$$

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$$= \sqrt{\frac{2}{\pi}} \left[0 - 0 + \int_0^\infty \frac{e^{-ax}}{a^2+K^2} (a \sin Kx + K \cos Kx)\, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+K^2} \int_0^\infty e^{-ax} \sin Kx\, dx + \sqrt{\frac{2}{\pi}} \frac{K}{a^2+K^2} \int_0^\infty e^{-ax} \cos Kx\, dx$$

Again using formul used in privious page

$$= \sqrt{\frac{2}{\pi}} \frac{a}{(a^2+K^2)} \left[\frac{e^{-ax}}{a^2+K^2} \{ -a \sin Kx - K \cos Kx \} \right]_0^\infty$$

$$+ \sqrt{\frac{2}{\pi}} \frac{K}{(a^2+K^2)} \left[\frac{e^{-ax}}{a^2+K^2} \{ -a \cos Kx + K \sin Kx \} \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{a}{a^2+K^2} \right) \left[0 + \frac{K}{a^2+K^2} \right] + \sqrt{\frac{2}{\pi}} \left(\frac{K}{a^2+K^2} \right) \left[0 + \frac{a}{a^2+K^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{aK}{(a^2+K^2)^2} + \sqrt{\frac{2}{\pi}} \frac{aK}{(a^2+K^2)^2}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2aK}{(a^2+K^2)^2} \right]$$

(b) Show that $\mathcal{F}_c\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}} \frac{a^2-K^2}{(a^2+K^2)^2}$

Solution:- Take L.H.S $= \mathcal{F}_c\{x.e^{-ax}\}$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty x \cdot e^{-ax} \cos Kx\, dx$$

Using formula

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2} \{ a \cos bx + b \sin bx \}$$

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2} \{ a \sin bx - b \cos bx \}$$

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integration by parts

$$=\sqrt{\frac{2}{\pi}}\left[\left\{x\ \frac{e^{-ax}}{a^2+K^2}\left(-a\cos Kx+K\sin Kx\right)\right\}_0^{\infty}-\right.$$

$$\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{e^{-ax}}{a^2+K^2}\left(-a\cos Kx+K\sin Kx\right)$$

$$=\sqrt{\frac{2}{\pi}}\left(0-0\right)+\sqrt{\frac{2}{\pi}}\ \frac{a}{a^2+K^2}\int_0^{\infty}e^{-ax}\cos Kx\,dx$$

$$-\sqrt{\frac{2}{\pi}}\ \frac{K}{a^2+K^2}\int_0^{\infty}e^{-ax}\sin Kx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\ \frac{a}{a^2+K^2}\left[\frac{e^{-ax}}{a^2+K^2}\left\{-a\cos Kx+K\sin Kx\right\}_0^{\infty}\right.$$

$$-\sqrt{\frac{2}{\pi}}\ \frac{K}{a^2+K^2}\left[\frac{e^{-ax}}{a^2+K^2}\left\{-a\sin Kx-K\cos Kx\right\}_0^{\infty}\right.$$

$$=\sqrt{\frac{2}{\pi}}\ \frac{a}{(a^2+K^2)^2}\left(0+a\right)+\sqrt{\frac{2}{\pi}}\ \frac{K}{(a^2+K^2)^2}\left(0-K\right)$$

$$=\sqrt{\frac{2}{\pi}}\ \frac{a^2}{(a^2+K^2)^2}-\sqrt{\frac{2}{\pi}}\ \frac{K^2}{(a^2+K^2)^2}$$

Written by
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$$=\sqrt{\frac{2}{\pi}}\left[\frac{a^2-K^2}{(a^2+K^2)^2}\right]$$

* * ⟷ * *

Example:- Calculate Fouxier Sin transfoom of $f(x)=e^{-x}\cos x$

Solution:- $F_s\{e^{-x}\cos x\}=\sqrt{\frac{2}{\pi}}\int_0^{\infty}e^{-x}\cos x\sin Kx\,dx$

$$=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{e^{-x}}{2}\left(2\sin Kx\cdot\cos x\right)dx$$

As we know $2\sin\alpha\cos\beta=\sin(\alpha+\beta)+\sin(\alpha-\beta)$

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$$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \{\sin(K+1)x + \sin(K-1)x\} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-x} \sin(K+1)x \, dx + \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-x} \sin(K-1)x \, dx \quad (i)$$

$\downarrow I_1$ $\qquad\qquad$ $\downarrow I_2$

Take $I_1 = \int_0^\infty e^{-x} \sin(K+1)x \, dx$

using the formula

$$\int e^{ax} \sin bx = \frac{e^{ax}}{a^2+b^2} [a \sin bx - b \cos bx]$$

$$I_1 = \left[\frac{e^{-x}}{(-1)^2+(K+1)^2} \{-\sin(K+1)x - (K+1)\cos(K+1)x\}\right]_0^\infty$$

$$I_1 = 0 - \frac{1}{1+(K+1)^2} [0 - (K+1)]$$

$$I_1 = \frac{(K+1)}{1+(K+1)^2} = \frac{K+1}{1+K^2+1+2K} = \frac{K+1}{K^2+2K+2}$$

Now Take $I_2 = \int_0^\infty e^{-x} \sin(K-1)x \, dx$

$$I_2 = \left[\frac{e^{-x}}{(-1)^2+(K-1)^2} \{\sin(K-1)x - (K-1)\cos(K-1)x\}\right]_0^\infty$$

$$I_2 = 0 - \frac{1}{1+(K-1)^2} [0 - (K-1)]$$

$$I_2 = \frac{(K-1)}{1+(K-1)^2}$$

$$I_2 = \frac{K-1}{1+K^2-2K+1}$$

$$I_2 = \frac{K-1}{K^2-2K+2}$$

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putting the results of I_1 and I_2 in (1)

$$F_s\{e^{-x}\cos x\} = \frac{1}{\sqrt{2\pi}}\frac{K+1}{K^2+2K+2} + \frac{1}{\sqrt{2\pi}}\cdot\frac{K-1}{K^2-2K+2}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{(K+1)(K^2-2K+2)+(K-1)(K^2+2K+2)}{(K^2+2K+2)(K^2-2K+2)}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{K^3-2K^2+2K+K^2-2K+2+K^3+2K^2+2K-K^2-2K-2}{(K^2+2+2K)(K^2+2-2K)}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2K^3}{(K^2+2)^2-(2K)^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2K^3}{K^4+4+4K^2-4K^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{K^3}{K^4+4}\right]$$

Written by
Labeeb Ahmad

5/2010

Question:- Find the Fourier Cosine transform of $f(x) = e^{-x}\cos x$

Solution:-

$$F\{e^{-x}\cos x\} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos x\cos Kx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-x}}{2}\left[2\cos Kx\cdot\cos x\right]dx$$

As we know that $2\cos\alpha\cdot\cos\beta = \cos(\alpha+\beta)+\cos(\alpha-\beta)$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\left[\cos(K+1)x+\cos(K-1)x\right]dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-x}\cos(K+1)x\,dx + \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-x}\cos(K-1)x\,dx$$

(1)

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Using formula $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\{a\cos bx + b\sin bx$

Let $I_1 = \int_0^\infty e^{-x}\cos(K+1)x\,dx$

$$I_1 = \left[\frac{e^{-x}}{(-1)^2+(K+1)^2}\{-\cos(K+1)x + (K+1)\sin(K+1)x\}\right]_0^t$$

$$I_1 = 0 - \frac{1}{1+(K+1)^2}[-1+0]$$

$$I_1 = \frac{1}{K^2+2K+2}$$

Now Take $I_2 = \int_0^\infty e^{-x}\cos(K-1)x\,dx$

$$I_2 = \left[\frac{e^{-x}}{(-1)^2+(K-1)^2}\{-\cos(K-1)x + (K-1)\sin(K-1)x\}\right]$$

$$I_2 = 0 - \frac{1}{1+K^2-2K+1}(-1+0)$$

$$I_2 = \frac{1}{K^2-2K+2}$$

Putting the results of I_1 and I_2 in (1)

$$F_c\{e^{-x}\cos x\} = \frac{1}{\sqrt{2\pi}}\left[\frac{1}{K^2+2K+2}\right] + \frac{1}{\sqrt{2\pi}}\left[\frac{1}{K^2-2K+2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{K^2-2K+2+K^2+2K+2}{(K^2+2)^2-(2K)^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2K^2+4}{K4+4+4K^2-4K^2}\right]$$

$$= \frac{2}{\sqrt{2\pi}}\left[\frac{K^2+2}{K^2+4}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{K^2+2}{K^2+4}\right]$$

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(58)

Solution of D-E using Fourier Transform

* * * *

A/2016, S/2010

Question 1:- Use Fourier transform method to obtain a unique solution of the initial / boundary value problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad -\infty < x < \infty, \; t > 0$$

$$u(x,0) = e^{-\alpha x^2}, \; u(x) \text{ and } u_x(x) \to 0 \text{ as } x \to \pm\infty$$

Solution:- The problem describes the flow of heat in a linear medium of infinite dimentions, the initial temprature distribution is given by the sharply peaked Gaussian function

$$u(x,0) = e^{-\alpha x^2}$$

Physicelly it implies that if the rod is read from the middle, the temprature at end points is very very small.

Consider

$$U_{xx}(x,t) = U_t(x,t) \qquad (1)$$

Taking fourier transform on both sides

$$\mathcal{F}\{U_{xx}(x,t)\} = \mathcal{F}\{U_t(x,t)\}$$

$$(-iK)^2 \mathcal{F}\{U(x,t)\} = \frac{d}{dt}\mathcal{F}\{U(x,t)\}$$

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$$-K^2 U(K,t) = \frac{d}{dt} U(K,t)$$

$$\frac{d\,U(K,t)}{U(K,t)} = -K^2 . dt$$

Integrating on both sides w.r.t "t"

$$\ln U(K,t) = -K^2 t + C$$

$$U(K,t) = A(K) e^{-K^2 t} \quad \text{——} (2)$$

where $A(K)$ is a constant of integration

Now we use the Boundary condition

$$U(x,0) = e^{-\alpha x^2} \longrightarrow \text{(given)}$$

$$\mathcal{F}\{U(x,0)\} = \mathcal{F}\{e^{-\alpha x^2}\}$$

$$U(K,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{2iKx} \cdot e^{-\alpha x^2} \cdot dx$$

$$U(K,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha\left(x^2 - \frac{2iKx}{\alpha}\right)} \cdot dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha\left[x^2 - \frac{2iKx}{\alpha} + \left(\frac{iK}{\alpha}\right)^2 - \left(\frac{iK}{\alpha}\right)^2\right]} \cdot dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha\left[\left(x - \frac{iK}{\alpha}\right)^2 + \frac{K^2}{4\alpha^2}\right]} \cdot dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left\{\sqrt{\alpha}\left(x - \frac{iK}{2\alpha}\right)\right\}^2} \cdot e^{\frac{-\alpha K^2}{4\alpha^2}} \cdot dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{-K^2}{4\alpha}} \int_{-\infty}^{\infty} e^{-\left[\sqrt{\alpha}\left(x - \frac{iK}{2\alpha}\right)\right]^2} \cdot dx$$

Let t $\quad \sqrt{\alpha}\left(x - \frac{iK}{2\alpha}\right) = t$

$$dx = \frac{1}{\sqrt{\alpha}} \cdot dt$$

$$U(K,0) = \frac{1}{\sqrt{2\pi}} e^{\frac{-K^2}{4\alpha}} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{\sqrt{\alpha}}$$

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$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{\alpha}} e^{-k^2/4\alpha} \int_{-\infty}^{\infty} e^{-t^2} \cdot dt$$

$$= \frac{1}{\sqrt{2\pi\alpha}} e^{-\frac{k^2}{4\alpha}} \cdot \sqrt{\pi}$$

$$\therefore \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$$

$$U(K,0) = \frac{1}{\sqrt{2\alpha}} e^{-\frac{K^2}{4\alpha}} \quad \text{---(3)}$$

$$U(K,0) = A(K)$$

$$A(K) = \frac{1}{\sqrt{2\alpha}} e^{-\frac{K^2}{4\alpha}}$$

Written by
Labeeb Ahmad

subsitating in (2)

$$U(K,t) = \frac{1}{\sqrt{2\alpha}} e^{-\frac{K^2}{4\alpha}} \cdot e^{-K^2 \cdot t}$$

$$= \frac{1}{\sqrt{2\alpha}} e^{-K^2\left[\frac{1}{4\alpha} + t\right]}$$

$$= \frac{1}{\sqrt{2\alpha}} e^{-K^2\left[\frac{1+4\alpha t}{4\alpha}\right]}$$

Let $\frac{1+4\alpha t}{4\alpha} = \alpha'$

Then

$$U(K,t) = \frac{1}{\sqrt{2\alpha}} e^{-K^2\alpha'}$$

Taking inverse Fourier transform on both side

$$\mathcal{F}^{-1}\{U(K,t)\} = \mathcal{F}^{-1}\left\{\frac{1}{\sqrt{2\alpha}} e^{-K^2\alpha'}\right\}$$

$$F^{-1}\{U(K,t)\} = \frac{1}{\sqrt{2\alpha}} F^{-1}\{e^{-K^2\alpha'}\}$$

$$= \frac{1}{\sqrt{2\alpha}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iKx} \cdot e^{-K^2\alpha'} dK$$

$$= \frac{1}{2\sqrt{\alpha\pi}} \int_{-\infty}^{\infty} e^{-\alpha'\left[K^2 + \frac{iKx}{\alpha'}\right]} dK$$

$$= \frac{1}{2\sqrt{\pi\alpha}} \int_{-\infty}^{\infty} e^{-\alpha'\left[K^2 + \frac{iKx}{\alpha'} + \left(\frac{ix}{2\alpha'}\right)^2 - \left(\frac{ix}{2\alpha'}\right)^2\right]} dK$$

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$$\mathcal{F}^{-1}\{U(K,t)\} = \frac{1}{2\sqrt{\alpha\pi}} \int_{-\infty}^{\infty} e^{-\alpha'\left(\left[K+\frac{ix}{2\alpha'}\right]^2 + \frac{x^2}{4\alpha'^2}\right)} \cdot dK$$

$$= \frac{1}{2\sqrt{\alpha\pi}} \int_{-\infty}^{\infty} e^{-\alpha'\left(K+\frac{iK}{2\alpha'}\right)^2} \cdot e^{\frac{-\alpha' x^2}{4\alpha'^2}} \cdot dK$$

$$= \frac{1}{2\sqrt{\alpha\pi}} e^{\frac{-x^2}{4\alpha'}} \int_{-\infty}^{\infty} e^{-\alpha'\left(K+\frac{ix}{2\alpha'}\right)^2} \cdot dK$$

$$= \frac{1}{2\sqrt{\alpha\pi}} e^{-x^2/4\alpha'} \int_{-\infty}^{\infty} e^{-\left[\sqrt{\alpha'}\left(K+\frac{ix}{2\alpha'}\right)\right]^2} dK$$

Let $\sqrt{\alpha'}\left(K+\frac{ix}{2\alpha'}\right)^2 = \theta$

$\sqrt{\alpha'}\, dK = d\theta$

$dK = \frac{d\theta}{\sqrt{\alpha'}}$

$$= \frac{1}{2\sqrt{\alpha\pi}} \cdot e^{-x^2/4\alpha'} \int_{-\infty}^{\infty} e^{-\theta^2} \frac{d\theta}{\sqrt{\alpha'}}$$

$$\mathcal{F}^{-1}\{U(K,t)\} = \frac{1}{2\sqrt{\alpha\pi}} e^{-x^2/4\alpha'} \frac{1}{\sqrt{\alpha'}} \sqrt{\pi}$$

$$U(x,t) = \frac{1}{2\sqrt{\alpha}} \cdot \frac{1}{\sqrt{\alpha'}} e^{-x^2/4\alpha'}$$

$$= \frac{1}{2\sqrt{\alpha}} \cdot \frac{1}{\sqrt{\frac{1+4\alpha t}{4\alpha}}} e^{-x^2 \cdot \frac{1}{4\left[\frac{1+4\alpha t}{4\alpha}\right]}}$$

$$= \frac{1}{\sqrt{1+4\alpha t}} e^{\frac{-x^2\alpha}{1+4\alpha t}}$$

$$U(x,t) = \frac{1}{\sqrt{1+4\alpha t}} \operatorname{Exp}\left[\frac{-\alpha x^2}{1+4\alpha t}\right]$$

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1/2015, A/2013, A/2008

Question 2:- Solve $U_{xx}(x,t) = \frac{1}{c^2} U_{tt}(x,t)$ —— (1)

$$\left.\begin{aligned} U(x,0) &= P(x) \\ U_t(x,0) &= Q(x) \end{aligned}\right\} \text{—— (2)}$$

and $U \to 0$, $U_x \to 0$ as $x \to \pm\infty$ by F.T method.

Solution:-

Take equ (1), Taking Fourier transform of both side

$$\therefore \mathcal{F}\{U_{xx}(x,t)\} = \frac{1}{c^2}\mathcal{F}\{U_{tt}(x,t)\}$$

$$(-iK)^2 F\{U(x,t)\} = \frac{1}{c^2}\frac{d^2}{dt^2}\{U(K,t)\}$$

$$-K^2 U(K,t) = \frac{1}{c^2}\frac{d^2}{dt^2}U(K,t)$$

$$-c^2K^2 U(K,t) = \frac{d^2}{dt^2}U(K,t)$$

$$\frac{d^2}{dt^2}U(K,t) + c^2K^2 U(K,t) = 0$$

Associated equation is

$$D^2 + c^2K^2 = 0$$

$$D = \pm icK$$

$$U(K,t) = Ae^{icKt} + Be^{-icKt} \text{ —— (3)}$$

Now we use the Boundary conditions

From (2), by taking F.T, we have

$\mathcal{F}\{U(x,0)\} = \mathcal{F}\{P(x)\}$	$\mathcal{F}\{U_t(x,0)\} = \mathcal{F}\{Q(x)\}$
$U(K,0) = P(K)$	$\frac{d}{dt}U(K,0) = Q(K)$

From (3) $\Rightarrow$

$$U(K,0) = Ae^{0} + B^{0}$$

$$U(K,0) = A + B$$

$$P(K) = A + B$$

or $$A + B = P(K) \text{ —— (4)}$$

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Now Differentiate equ (3) w.r.t "t"

$$\frac{d}{dt}U(K,t) = AicKe^{icKt} - BicKe^{-icKt}$$

$$\frac{d}{dt}U(K,0) = icK[A-B] \quad \text{---} (5)$$

$$Q(K) = icK[A-B]$$

$$A-B = \frac{1}{icK}Q(K) \quad \text{---} (6)$$

Subtracting (4) and (6)

$$2B = P(K) - \frac{1}{icK}Q(K)$$

$$B = \frac{1}{2}\left[P(K) - \frac{1}{icK}Q(K)\right]$$

Adding (4) and (6)

$$2A = P(K) + \frac{1}{icK}Q(K)$$

$$A = \frac{1}{2}\left[P(K) + \frac{1}{icK}Q(K)\right]$$

putting these values in (3)

$$U(K,t) = \frac{1}{2}\left[P(K) + \frac{1}{icK}Q(K)\right]e^{icKt} + \frac{1}{2}\left[P(K) - \frac{1}{icK}Q(K)\right]e^{-ic}$$

$$U(K,t) = \frac{1}{2}P(K)\left[e^{icKt} + e^{-icKt}\right] + \frac{Q(K)}{icK}\left[e^{icKt} - e^{-icKt}\right]$$

Taking inverse F.T

$$\mathcal{F}^{-1}\{U(K,t)\} = \frac{1}{2}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left\{P(K)\left[e^{icKt} + e^{-icKt}\right]e^{-iKx}\right.$$

$$\left. + \frac{Q(K)}{icK}\left[e^{icKt} - e^{-icKt}\right]e^{-iKx}\right\}dK$$

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(264)

$$U(x,t) = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} P(K) \left[e^{iK[x-ct]} + e^{-iK[x+ct]} \right] dK$$

$$+ \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{Q(K)}{iK} \left[e^{-iK(x-ct)} - e^{iK(x+ct)} \right] dK$$

$$U(x,t) = \frac{1}{2}\{P(x-ct) - P(x+ct)\} + \frac{1}{2} I_0 \quad \text{---(7)}$$

where $I_0 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{Q(K)}{iK} \{ e^{-iK(x-ct)} - e^{-iK(x+ct)} \} dK$ ---(8)

Using the definition of inverse F.T, we have

$$V(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iKx} Q(K) dx \quad \text{---(9)}$$

integrating both side w.r.t "x" from $(x-ct)$ to $(x+ct)$, we obtain

$$\int_{x-ct}^{x+ct} V(x) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} Q(K) \int_{x-ct}^{x+ct} e^{-iKx} dx\, dK$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} Q(K) \left\{ \frac{e^{-iK(x+ct)} - e^{-iK(x-ct)}}{-iK} \right\} dK$$

$$\int_{x-ct}^{x+ct} V(x) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{Q(K)}{iK} \left\{ e^{-iK(x-ct)} - e^{-iK(x+ct)} \right\} dK$$

Multiplying both side by $\frac{1}{c}$, we get

$$\int_{x-ct}^{x+ct} \frac{V(x)\,dx}{c} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{Q(K)}{iCKt} \left\{ e^{-iK(x-ct)} - e^{-iK(x+ct)} \right\} dK \quad \text{---(10)}$$

From (7), (9) and (10), we have

$$f(x,t) = \frac{1}{2}\{P(x-ct) + P(x+ct)\} = \frac{1}{2c} \int_{x-ct}^{x+ct} V(x) dx$$

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A/2014

Question 3 :- Use Fourier transform method to solve the following problems

$$\frac{\partial^4 u}{\partial x^4} = \frac{1}{4}\frac{\partial^2 u}{\partial t^2} \quad \text{—— (1)}$$

$$u(x,0) = f(x) \quad , u_t(x,0) = 2g'(x) \quad \text{—— (2)}$$

Solution :- From (1) taking F.T on both sides

$$\mathcal{F}\left\{\frac{\partial^4 u(x,t)}{\partial x^4}\right\} = \mathcal{F}\left\{\frac{1}{4} u_{tt}(x,t)\right\}$$

$$(-2iK)^4 U(K,t) = \frac{1}{4}\frac{\partial^2}{\partial t^2} U(K,t) \quad \text{—— (3)}$$

$$K^4 U(K,t) = \frac{1}{4}\frac{\partial^2}{\partial t^2} U(K,t)$$

$$4K^4 U(K,t) = \frac{\partial^2}{\partial t^2} U(K,t)$$

$$\frac{\partial^2}{\partial t^2} U(K,t) - 4K^4 U(K,t) = 0$$

$$\left[D_t^2 - 4K^4\right] U(K,t) = 0$$

Associated equation is

$$D_t^2 - 4K^4 = 0$$

$$D_t = \pm 2K^2$$

Therefor $U(K,t) = Ae^{aK^2 t} + Be^{-aK^2 t}$ —— (4)

We now transform the initial conditions in terms of K by taking F.T of both sides of condition in (2)

$U(x,0) = f(x)$

$F(U(x,0)) = F\{f(x)\}$

$U(K,0) = F(K)$

$U_t(x,0) = 2g'(x)$

Taking F.T

$\frac{d}{dt} U(K,0) = 2(-iK)G(K)$

$= -2iKG(K)$

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from (4) $\Rightarrow U(K,0) = Ae^{0} + Be^{0}$

$$F(K) = A + B \quad \text{---} (6)$$

Again from (4) $\Rightarrow$

$$\frac{d}{dt} U(K,t) = 2K^2 Ae^{2K^2 t} - 2K^2 Be^{-2K^2 t}$$

$$\frac{d}{dt} U(K,0) = 2K^2 Ae^{0} - 2K^2 Be^{0}$$

$$= 2K^2 [A - B]$$

$$\Rightarrow iK\,G(K) = 2K^2 [A - B]$$

$$-\frac{i}{K} G(K) = A - B \quad \text{---} (7)$$

To find A and B we adding and subtracting (6), (7)

$$A = \frac{1}{2}\left\{ F(K) - \frac{i}{K} G(K) \right\}$$

$$B = \frac{1}{2}\left\{ F(K) + \frac{i}{K} G(K) \right\}$$

Written by
Labeeb Ahmad

Substituting in (4) and get

$$U(K,t) = \frac{1}{2}\left\{ F(K) - \frac{i}{K} G(K) \right\} e^{2K^2 t} + \frac{1}{2}\left\{ F(K) + \frac{i}{K} G(K) \right\} e^{-2K^2 t}$$

$$U(K,t) \equiv \frac{1}{2} F(K)\left\{ e^{2K^2 t} + e^{-2K^2 t} \right\} - \frac{1}{2}\frac{i}{K} G(K)\left\{ e^{2K^2 t} - e^{-2K^2 t} \right\}$$

Taking inverse transform, we have

$$U(x,t) = \mathcal{F}^{-1}\{U(K,t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iKx} U(K,t)\, dK$$

$$U(x,t) = \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(K)\left[e^{2K^2 t} + e^{-2K^2 t} \right] e^{-iKx} dK$$

$$-\frac{i}{K} \cdot \frac{1}{2\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} G(K)\left[e^{2K^2 t} - e^{-2K^2 t} \right] e^{-iKx} dK$$

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$$U(x,t) = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} F(K) \left[e^{-iK(x+i2Kt)} + e^{-iK(x-i2Kt)} \right] dK$$

$$- \frac{i}{2K\sqrt{2\pi}} \int_{-\infty}^{\infty} G(K) \left[e^{-iK(x+i2Kt)} - e^{-iK(x-i2Kt)} \right]$$

$$U(x,t) = \frac{1}{2} \left[f(x+i2Kt) + f(x-i2Kt) \right.$$

$$\left. - \frac{i}{2K} \left[g(x+i2Kt) - g(x-i2Kt) \right] \right]$$

Question 4:- $U_{xxxx} = \frac{1}{a^2} U_{tt}$ —— (1)

$U(x,0) = f(x)$, $U_t(x,0) = a g'(x)$ —— (2)

Solution:- Taking F.T of both sides of (1)

$$F\{U_{xxxx}\} = \frac{1}{a^2} F\{U_{tt}\}$$

$$(-2K)^4 U(K,t) = \frac{1}{a^2} \frac{d^2}{dt^2} U(K,t)$$

$$K^4 \cdot a^2 U(K,t) = \frac{d^2}{dt^2} U(K,t)$$

$$\frac{d^2}{dt^2} U(K,t) - a^2 K^4 U(K,t) = 0$$

$$(D^2 - a^2 K^4) U(K,t) = 0$$

The associated equation is

$$D^2 - a^2 K^4 = 0 \Rightarrow D = \pm aK^2$$

The general solution is

$$U(K,t) = Ae^{aK^2 t} + Be^{-aK^2 t} \quad \text{—— (4)}$$

From (2) $\Rightarrow F\{U(x,0)\} = F\{f(x)\}$

$$U(K,0) = F(K)$$

And $U_t(x,0) = a g'(x)$

Take F.T

$$\frac{d}{dt} U(K,0) = a(-iK) G(K)$$

$$= -aiK G(K)$$

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From ④ $\Rightarrow U(K,0) = Ae^{0} + Be^{0} = A+B$

$$F(K) = A+B \quad \text{---} \quad (5)$$

Again from ④ $\Rightarrow \frac{d}{dt} U(K,t) = aK^2 Ae^{aK^2 t} - aK^2 Be^{-aK^2 t}$

$$\frac{d}{dt} U(K,0) = aK^2 A^{0} - aK^2 Be^{0}$$

$$-\alpha 2K G_1(K) = aK^2 [A-B]$$

$$-\frac{2}{K} G_1(K) = A-B \quad \text{---} \quad (6)$$

Solving ⑤ & ⑥

$$A = \frac{1}{2}\left[F(K) - \frac{2}{K} G_1(K)\right]$$

$$B = \frac{1}{2}\left[F(K) + \frac{2}{K} G_1(K)\right]$$

Substituting these values in ④ we get

$$U(K,t) = \frac{1}{2}\left[F(K) - \frac{2}{K} G_1(K)\right] e^{aK^2 t} + \frac{1}{2}\left[F(K) + \frac{2}{K} G_1(K)\right] e^{-aK^2 t}$$

$$= \frac{1}{2} F(K)\left[e^{aK^2 t} + e^{-aK^2 t}\right] - \frac{2}{2K} G_1(K)\left[e^{aK^2 t} - e^{-aK^2 t}\right]$$

Taking inverse Fourier transform, we have

$$U(x,t) = \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(K)\left[e^{aK^2} + e^{-aK^2 t}\right] e^{-2Kx}\, dK$$

$$-\frac{2}{K} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} G_1(K)\left[e^{aK^2 t} - e^{-aK^2 t}\right] e^{-2Kx}\, dK$$

$$U(x,t) = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} F(K)\left[e^{-2K(x+2aKt)} + e^{-2K(x-2aKt)}\right] dK$$

$$\frac{-2}{2K\sqrt{2\pi}} \int_{-\infty}^{\infty} G_1(K)\left[e^{-2K(x+2aKt)} - e^{-2K(x-2aKt)}\right] dK$$

$$U(x,t) = \frac{1}{2}\left[f(x+2aKt) + f(x-2aKt)\right]$$

$$-\frac{2}{2K}\left[g(x+2aKt) - g(x-2aKt)\right]$$

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GREEN'S FUNCTIONS

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George Green (1793-1841) a British mathemati
Green's functions method is used for solving
Boundary value problems associated with
linear non-homogenous Differential Equations.

A,D 2009

Dirac Delta Function

In Science and mathematics, the Dirac delta function, or δ-function is a generalized function on the real line that is zero everywhere except at zero. The delta function is sometimes thought of as a hypothetical function whose graph is an infinitely high, infinitely thin spike at the origin, with total area one under the spike, and physically represents the density of an idealized point mass

The Dirac delta function is defined by the following properties

(i) $$\delta(x-t) = \begin{cases} \infty & ; \ x = t \\ 0 & ; \ x \neq t \end{cases}$$

(ii) $$\int_{-\infty}^{\infty} \delta(x-t)\,dx = 1$$

(iii) $$\int_{-\infty}^{\infty} \delta(x-t)\,f(x)\,dx = f(t)$$

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(27)

A/2011, A/2009

Green's Function associated with Regular S-L SYSTEM

$$\frac{d}{dx}\left[P(x)\cdot\frac{du}{dx}\right]+q(x)u(x)+\lambda r(x)u(x)=0$$

$$[P(x)u']'+q(x)u(x)+\lambda r(x)u(x)=0$$

OR

$$L(u)+\lambda r(x)u=0 \quad , \quad L=\frac{d}{dx}\left\{P(x)\frac{d}{dx}\right\}+q(x)$$

be the S-L equation with end conditions

$$\alpha_1 u(a)+\alpha_2 u'(a)=0$$

$$\beta_1 u(b)+\beta_2 u'(b)=0$$

which may also be written as

$$B_1(u)=0 \quad \text{and} \quad B_2(u)=0$$

where B is the Boundary operator.

Now it is a regular S-L system

$$B_1=\alpha_1+\alpha_2\frac{\partial}{\partial x} \quad , \quad B_2(u)=\beta_1+\beta_2\frac{\partial}{\partial x}$$

⇒ Under the assumption that $\lambda=0$ is not an eigen value of this system, the green function $G(x,t)$ associated with the system has the following properties.

(1) $G(x,t)$ considered as a function of x satisfies the D-E

$$L\{G(x,t)\}=0$$

in each of the subinterval $[a,t)$, and $[t,b]$

(2) $G(x,t)$ is continous for each value of x in the whole interval $[a,b]$

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i-e if we take limit as $x \to t$ for each piece of the solution in the above subinterval then the limit shaud be equal.

(3) $G(x,t)$ as a function of x satisfies the end conditions

$$B_1(G) = 0 \text{ , and } B_2(G) = 0$$

(4) $\frac{d}{dx}\{G(x,t)\} = G'(x,t)$ is discontinous at $x = t$ and more over

$$\lim_{x \to t^-} G'(x,t) - \lim_{x \to t^+} G'(x,t) = \frac{1}{p(t)}$$

or $$G'(t^-,t) - G'(t^+,t) = \frac{1}{p(t)}$$

EXERCISE

A/2011, S/2010

Question (1) ⇒ Construct the Green's function associated with the problem

$$u'' + \lambda u = 0 \text{ , } u(0) = 0 \text{ , } u(1) = 0$$

Solution :- Given

$u'' + \lambda u = 0$ which can be written as

$$(1 \cdot u')' + 0 \cdot u(x) + \lambda(1) \cdot u(x) = 0$$

Compare with $(p(x) \cdot u')' + q(x)u(x) + \lambda r(x) \cdot u(x) = 0$

S-L Equation

So Here $p(x) = 1 \Rightarrow p(t) = 1$

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i) First we check $\lambda = 0$ is an eigen value

For this put $\lambda = 0$

we have $u''(x) = 0$

Integrating both side $u'(x) = A$

gain integrating $u(x) = Ax + B$ —— (2)

Now using th bandary Conditions,

we get $u(0) = A(0) + B$

$0 = B \Rightarrow B = 0$

(2) becomes $u(x) = Ax$

$u(1) = A(1)$

$0 = A \Rightarrow A = 0$

Therefor $u = 0$ is the solution of the problem

corresponding to $\lambda = 0$

Thus $\lambda = 0$ is not an eigen value.

(ii) $G(x,t)$ satisfies the D-E

$$\frac{d^2}{dx^2}\left[G(x,t)\right] = 0$$ in each subinterval

$[0,t]$ and $]t,1]$, Therefor we have

$$G(x,t) = \begin{cases} Ax + B & ; \ 0 \le x \le t \\ Cx + D & ; \ t < x \le 1 \end{cases}$$

iii) $G(x,t)$ is continous everywhere and in particular at $x = t$

Therefor

$$At + B = Ct + D$$

$$At + B - Ct = D$$

OR $\Rightarrow$ $D = (A - C)t + B$

Hence $G(x,t)$ becomes

$$G(x,t) = \begin{cases} Ax + B & ; \ 0 \le x \le t \\ Cx + (A-C)t + B & ; \ t < x \le 1 \end{cases}$$

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(iv) $G(x,t)$ satisfies the end conditions

$$G(0,t) = 0$$

$$A(0)+B = 0 \quad \Rightarrow \quad \boxed{B=0}$$

And 2nd B.C $G(1,t) = 0$

$$C+(A-C)t+B = 0$$

$$C+At-Ct+0=0 \quad (\because B=0)$$

$$\text{So} \quad A = \frac{Ct-C}{t} = \frac{(t-1)C}{t}$$

Thus $G(x,t)$ becomes

$$G(x,t) = \begin{cases} \left(\frac{t-1}{t}\right)Cx & ; 0 \le x \le t \\ Cx+\left[\left(\frac{t-1}{t}\right)C-C\right]t+0 & ; t < x \le 1 \end{cases}$$

$$G(x,t) = \begin{cases} \left(\frac{t-1}{t}\right)Cx & ; 0 \le x \le t \\ Cx+\left[\frac{tC-C-tC}{t}\right]t & ; t < x \le \end{cases}$$

$$G(x,t) = \begin{cases} \left(\frac{t-1}{t}\right)Cx & ; 0 \le x \le t \\ Cx-C & ; t < x \le 1 \end{cases}$$

(v) $G(x,t)$ satisfies

$$\lim_{x \to t^-} (G'(x,t)) - \lim_{x \to t^+} G'(x,t) = \frac{1}{p(t)}$$

$$G'(x,t) = \begin{cases} \left(\frac{t-1}{t}\right)C & ; 0 \le x \le t \\ C & ; t < x \le 1 \end{cases}$$

Now

$$\lim_{x \to t^-} \left[\left(\frac{t-1}{t}\right)C\right] - \lim_{x \to t^+} [C] = \frac{1}{1} \quad (\because p(t) =$$

$$\left(\frac{t-1}{t}\right)C - C = 1$$

$$\frac{Ct-C-Ct}{t} = 1$$

$$-C = t \quad \Rightarrow \quad \boxed{C = -t}$$

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Therefore $G_1(x,t)$ becomes

$$G_1(x,t) = \begin{cases} \dfrac{(t-1)(-t)x}{t} & ; 0 < x \leq t \\ -tx - (-t) & ; t < x \leq 1 \end{cases}$$

Finally

$$G(x,t) = \begin{cases} (1-t)x & ; \quad 0 \leq x \leq t \\ (1-x)t & ; \quad t < x \leq 1 \end{cases}$$

A/2013, A/2000

Question 2:- Find Green's function associated with problem

$$u''(x) + \lambda u(x) = 0$$

$$u'(0) = 0 \quad , \quad u(1) = 0$$

Solution:- Given $u''(x) + \lambda u(x) = 0$ ——— (1)

Comparing with S-L equation

we get $p(x) = 1 \Rightarrow p(t) = 1$

To construct Green's function $G(x,t)$ we first check if $\lambda = 0$ is an eigen value

(i) put $\lambda = 0$ in (1) we get

$$u'' = 0$$

integrating both side $u' = A$

gain integrating $u = Ax + B$ ——— (2)

Now applying Boundary Conditions

$u'(x) = 0 \Rightarrow A = 0$

(2) becomes $u = B$

apply 2nd B.C $u(1) = 0 \Rightarrow B = 0$

Hence we obtain the trivial solution

So $\lambda = 0$ is not eigen value.

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(ii) $G(x,t)$ regarded as a function of x satisfies the D-E $G''(x,t) = 0$ in each subinterval $[0,t]$ and $]t,1]$.

Therefore

$$G(x,t) = \begin{cases} Ax + B & ; \ 0 \leq x \leq t \\ Cx + D & ; \ t < x \leq 1 \end{cases}$$

(iii) $G(x,t)$ as a function of x satisfies the Boundary conditions.

$$G'(0,t) = 0 \Rightarrow A = 0$$

And

$$G(1,t) = 0$$

$$C(1) + D = 0 \Rightarrow C = -D$$

Thus $G(x,t)$ becomes

$$G(x,t) = \begin{cases} B & ; \ 0 \leq x \leq t \\ -Dx + D & ; \ t < x \leq 1 \end{cases}$$

(iv) $G(x,t)$ as a function of x is continous at all points particular at $x = t$

$$B = -Dt + D$$

$$B = (1-t)D$$

Now $G(x,t)$ becomes

$$G(x,t) = \begin{cases} D(1-t) & ; \ 0 \leq x \leq t \\ D(1-x) & ; \ t < x \leq 1 \end{cases}$$

Differentiate w.r.t "x" $G'(x,t) = \begin{cases} 0 & ; \ 0 \leq x \leq t \\ -D & ; \ t < x \leq 1 \end{cases}$

(v) $G(x,t)$ satisfies

$$\lim_{x \to t^-} G'(x,t) - \lim_{x \to t^+} G'(x,t) = 1/p(t)$$

$$0 - (-D) = 1/1$$

$$D = 1$$

Hence $G(x,t)$ becomes

$$G(x,t) = \begin{cases} (1-t) & ; \ 0 \leq x \leq t \\ (1-x) & ; \ t < x \leq 1 \end{cases}$$

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V2010, A/2009, S/2009, A/1994, A/1992, A/1991

Question 3:- Find Green's function associated with Problem

$x u'' + u' + \lambda \gamma(x) u = 0$ with

$u(0)$ is finite and $u(1) = 0$

Solution:- Given D-E

$$x u'' + u' + \lambda \gamma(x) u = 0 \quad \text{—(i)}$$

Re-write equation (i) in S-L form

$$(x u')' + \lambda \gamma(x) u = 0$$

Here $P(x) = x \neq 0$

Take $\lambda = 0$ in equation (i)

$$x u'' + u' = 0$$

$$(x u')' = 0$$

integrating both sides $x u' = A$

$$u' = \frac{A}{x}$$

again integrating both side

$$u = A \ln x + B$$

Apply the boundary conditions

$$u(1) = 0$$

$$A \ln(1) + B = 0$$

$$0 + B = 0 \Rightarrow \boxed{B = 0}$$

Now apply 2nd B.Condition $u(0)$ is finite

$$u(0) = A \ln(0) + B$$

$$= A \ln(0) + 0$$

$= A \ln(0)$ only finite, iff $\boxed{A = 0}$

For $\lambda = 0$, Thus solution is trivial

Therefor $\lambda = 0$ is not an eigen value of the given system.

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Now we can construct Green's function associated with the boundary value poablem.

(i) $G(x,t)$ as a function of x satisfies the given D-E

So $x\,G'' + G' = 0$

in each subinterval $[0,t]$ and $(t,1]$

Therefor we can write

$$G(x,t) = \begin{cases} A\ln x + B & ; 0 \leq x \leq t \\ c\ln x + D & ; t < x \leq 1 \end{cases}$$

(ii) $G(x,t)$ as a function of x satisfies B.C,s

$G(0,t) = A\ln(0) + B$ only

finite iff $\boxed{A = 0}$

$G(1,t) = C\ln(1) + D$

$C(0) + D = 0 \Rightarrow \boxed{D = 0}$

Thus $G(x,t)$ becomes

$$G(x,t) = \begin{cases} B & ; 0 \leq x \leq t \\ c\ln x & ; t < x \leq 1 \end{cases}$$

(iii) $G(x,t)$ is countinous at $x = t$

So $B = C\ln(t)$

Now $G(x,t)$ becomes

$$G(x,t) = \begin{cases} C\ln(t) & ; 0 \leq x \leq t \\ C\ln(x) & t < x \leq 1 \end{cases}$$

(iv) $G'(x,t)$ is discontinuous at $x = t$. Moreove satisfy

$$\lim_{x \to t^-} G'(x,t) - \lim_{x \to t^+} G'(x,t) = \frac{1}{P(t)}$$

$$0 - \frac{c}{t} = \frac{1}{t} \Rightarrow \boxed{C = -1}$$

Hence $G(x,t)$ becomes

$$G(x,t) = \begin{cases} -\ln t & ; 0 \leq x \leq t \\ -\ln x & ; t < x \leq 1 \end{cases}$$

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(5.78)

A/2012, A/2008, A/2006, A/2000, A/1991

Question 4 :- Construct Green's function for B.V.P

$$x u'' + u' - \frac{n^2}{x} u + \lambda r(x) u = 0$$

$u(0)$ is finite and $u(1) = 0$

Solution :- The given D-E is

$$x u'' + u' - \frac{n^2}{x} u + \lambda r(x) u = 0 \quad \text{---(i)}$$

Re-write equation (i) in S-L form

$$(x u')' - \frac{n^2}{x} u + \lambda r(x) = 0$$

Here $p(x) = x \neq 0$ $\quad x \neq 0$ and $q(x) = -\frac{n^2}{x}$

To Construct Green's function we first check if $\lambda = 0$ is eigen value put $\lambda = 0$

$$x u'' + u' - \frac{n^2}{x} = 0$$

$$x^2 u'' + x u' - n^2 = 0 \quad \text{---(ii)}$$

It is cauchy Euler Equation To solve it

put $x = e^t$ put $x = e^t$

or $t = \ln x$

$$\frac{dt}{dx} = \frac{1}{x}$$

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As $\frac{du}{dx} = \frac{du}{dt} \cdot \frac{dt}{dx}$

$$\frac{du}{dx} = \frac{du}{dt} \left(\frac{1}{x}\right) \quad \text{---(iii)}$$

$$D u = \frac{1}{x} \cdot \Delta u \quad \therefore D = \frac{d}{dx} \text{ and } \Delta = \frac{d}{dt}$$

$$\rightarrow \quad x D = \Delta$$

Again Diffeerentiate equation (iii)

$$\frac{d^2 u}{dx^2} = \frac{d}{dx}\left[\frac{1}{x} \cdot \frac{du}{dt}\right]$$

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$$\frac{d^2u}{dx^2} = -\frac{1}{x^2}\cdot\frac{du}{dt} + \frac{1}{x}\cdot\frac{d^2u}{dt^2}\cdot\frac{dt}{dx}$$

$$\frac{d^2u}{dx^2} = -\frac{1}{x^2}\frac{du}{dt} + \frac{1}{x}\cdot\frac{d^2u}{dt^2}\left[\frac{1}{x}\right]$$

$$D^2(u) = -\frac{1}{x^2}\Delta(u) + \frac{1}{x^2}\Delta^2(u)$$

$$D^2(u) = \frac{1}{x^2}\left(-\Delta + \Delta^2\right)u$$

$$\rightarrow \quad x^2D^2 = \Delta(\Delta-1)$$

putting these values in (ii)

$$(\Delta(\Delta-1) + \Delta - n^2)\,u = 0$$

$$(\Delta^2 - \Delta + \Delta - n^2)\,u = 0$$

$$(\Delta^2 - n^2)\,u = 0$$

Characteristic Equation is

$$m^2 - n^2 = 0 \Rightarrow m = \pm n$$

General Solution is

$$u(t) = Ae^{nt} + Be^{-nt}$$

$\therefore$ Since $e^t = x \Rightarrow u(x) = Ax^n + Bx^{-n}$

Apply the boundary conditions

$u(0)$ is finite

$$u(0) = A(0)^n + \frac{B}{(0)^n}$$

$$= 0 + \frac{B}{0} \text{ only finite iff } \boxed{B=0}$$

Now Second B.C $u(1) = 0 \Rightarrow A = 0$

For $\lambda = 0$ the solution is trivial

Therefore for $\lambda = 0$ is not an eigen value of given system.

Now we construct the Green's function associated with B.C's.

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(i) $G(x,t)$ satisfies the D-E $x G''(x,t) + G'(x,t) - \frac{n^2}{x} G = 0$ in each subinterval $[0,t]$ and $]t,1]$

Therefor we can write

$$G(x,t) = \begin{cases} A x^n + B x^{-n} ; & 0 \le x \le t \\ C x^n + D x^{-n} ; & t < x \le 1 \end{cases}$$

(ii) $G(x,t)$ regarded as function of x satisfies the given Boundary Conditions

$G(0,t)$ is finite $\Rightarrow B = 0$

$G(1,t) = 0 \Rightarrow C + D = 0 \Rightarrow C = -D$

$G(x,t)$ becomes

$$G(x,t) = \begin{cases} A x^n ; & 0 \le x \le t \\ C(x^n) - C(x^{-n}) ; & t < x \le 1 \end{cases}$$

$$G(x,t) = \begin{cases} A x^n ; & 0 \le x \le t \\ C\left[x^n - x^{-n}\right] ; & t < x \le 1 \end{cases}$$

(iii) $G(x,t)$ regarding as a function of x is continuous at $x = t$

$$A t^n = C(t^n - t^{-n})$$

$$A = \frac{C(t^n - t^{-n})}{t^n}$$

$$A = C(1 - t^{-2n})$$

Thus

$$G(x,t) = \begin{cases} C x^n (1 - t^{-2n}) ; & 0 \le x \le t \\ C(x^n - x^{-n}) ; & t < x \le 1 \end{cases}$$

(iv) $G'(x,t)$ is discontinous at $x = t$

More over satisfies

$$\lim_{x \to t^+} G'(x,t) - \lim_{x \to t^-} G'(x,t) = \frac{-1}{P(t)}$$

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$$C[nt^{n-1} + nt^{-n-1}] - Cnt^{n-1}[1 - t^{-2n}] = \frac{-1}{t}$$

$$nCt^{n-1} + nCt^{-n-1} - nCt^{n-1} + nCt^{-n-1} = \frac{-1}{t}$$

$$2nCt^{-n-1} = \frac{-1}{t}$$

$$C = \frac{-t^n}{2n}$$

Hence $G(x,t)$ becomes

$$G(x,t) = \begin{cases} \left[\frac{-1}{2n}\right][t^n - t^{-n}]x^n ; & 0 \le x \le t \\ \left[\frac{-1}{2n}\right][x^n - x^{-n}]t^n ; & t < x \le 1 \end{cases}$$

A/2010

Question 5 :- Find Green's function associated with the Problem

$$u''(x) + \lambda u(x) = 0$$

$$u'(0) = 0 \quad , \quad u(2) = 0$$

Solution :- The given D-E is

$$u'' + \lambda u = 0 \quad \text{---(i)}$$

Re - write the D-E (i) in S-L form

$$(1 \cdot u')' + \lambda u = 0$$

Here $P(x) = 1 \neq 0 \Rightarrow P(t) = 1$

To construct the green's function $G(x,t)$ we first check if $\lambda = 0$ is an eigen value

(i) put $\lambda = 0$ in (i) we get

$$u'' = 0$$

integrating both side $u' = A$

again integrating $u = Ax + B$

Now apply boundary Condition

$$u'(x) = A$$

$$0 = A \Rightarrow \boxed{A = 0}$$

$u(x)$ becomes $u(x) = B$

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$u(2) = 0 \Rightarrow \boxed{B = 0}$

Hence we obtain the trivial solution so $\lambda = 0$ is not eigen value

(ii) $G(x,t)$ regarded as a function of x satisfies the D-E $G_7''(x,t) = 0$ in each subinterval $[0,t]$ and $]t,1]$

Therefor $G_7(x,t) = \begin{cases} Ax + B & ; \ 0 \leq x \leq t \\ Cx + D & ; \ t < x \leq 2 \end{cases}$

(iii) $G(x,t)$ regarding as the function of x satisfing the boundary conditions

Differentiate $G(x,t)$

for apply B.Cs $G'(x,t) = \begin{cases} A & ; \ 0 \leq x \leq t \\ C & ; \ t < x \leq 2 \end{cases}$

$G'(0,t) = 0 \Rightarrow A = 0$

$G(2,t) = 0$

$C(2) + D = 0 \Rightarrow D = -2C$

so $G(x,t)$ becomes

$G(x,t) = \begin{cases} B & ; \ 0 \leq x \leq t \\ Cx - 2C & ; \ t < x \leq 2 \end{cases}$

(iv) $G(x,t)$ as a function of x is countinous everywhere in $(0,2]$ and in particular at $x = t$

$B = Ct - 2C$

$B = C[t-2]$

Now $G(x,t)$ becomes

$G(x,t) = \begin{cases} C[t-2] & ; \ 0 \leq x \leq t \\ C(x-2) & ; \ t < x \leq 2 \end{cases}$

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$$G_1'(x,t) = \begin{cases} 0 & ; \quad 0 \le x \le t \\ c & ; \quad t < x \le 2 \end{cases}$$

(v) $G'(x,t)$ is discontinuous at $x = t$

Moreover satisfies the discontinuity Conditio

$$\lim_{x \to t^-} G_1'(x,t) - \lim_{x \to t^+} G_1'(x,t) = \frac{1}{p(t)}$$

$$0 - c = 1$$

$$\boxed{c = -1}$$

$G_1(x,t)$ becomes

$$G_1(x,t) = \begin{cases} -1(t-2) & ; \quad 0 \le x \le t \\ -1(x-2) & ; \quad t < x \le 2 \end{cases}$$

$$G_1(x,t) = \begin{cases} (2-t) & ; \quad 0 \le x \le t \\ (2-x) & ; \quad t < x \le 2 \end{cases}$$

A/2007 , A/2001 , A/1991

Question 6 :- Construct the Green's function associated with problem

$$u'' + \lambda u = 0 \quad ; \; u(0) + u'(1) = 0$$

$$u(1) + 2u'(0) = 0$$

Solution :- The given D-E is

$$u'' + \lambda u = 0 \quad \text{——(i)}$$

Re-write equation (i) in S-L form

$$(1 \cdot u')' + \lambda u = 0$$

Here $p(x) = 1 \neq 0 \Rightarrow p(t) = 1$

To construct the green's function we first check if $\lambda = 0$ is an eigen value

put $\lambda = 0$ in (i)

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$$u''(x) = 0$$

Integrating both side $u'(x) = A$

again integrating both side $u(x) = Ax + B$

Now apply boundary conditions

$$u(0) + u'(1) = 0$$

$$B + A = 0 \quad \text{——(ii)}$$

$$u(1) + 2u'(0) = 0$$

$$[A+B] + 2[A] = 0$$

$$3A + B = 0 \quad \text{——(iii)}$$

Subtracting (2) from (iii)

$$3A + B = 0$$

$$\underline{A + B = 0}$$

$$2A = 0$$

$$\boxed{A = 0}$$

From (ii) put $A = 0$ $\boxed{B = 0} \Rightarrow u(x) = 0$

for $\lambda = 0$, the solution is trivial

Therefor $\lambda = 0$ is not an eigen value of the given system.

(ii) $G(x,t)$ regarded as a function of x satisfies the D-E $G''(x,t) = 0$

As
$$G(x,t) = \begin{cases} Ax + B \; ; \; 0 \le x \le t \\ Cx + D \; ; \; t < x \le 1 \end{cases}$$

(iii) $G(x,t)$ satisfies the boundary condition

$$G(0,t) + G'(1,t) = 0$$

$$B + C = 0$$

$$B = -C$$

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$$G(1,t) + 2G'(0,t) = 0$$

$$[C+D] + 2[A] = 0$$

$$D = -C - 2A$$

Now $G(x,t)$ becomes

$$G(x,t) = \begin{cases} Ax - C & ; 0 \le x \le t \\ Cx - C - 2A & ; t < x \le 1 \end{cases}$$

(iv) $G(x,t)$ regarded as function of x is countinuus everywhere in $[0,1]$, paticularly $x =$

$$At - C = Ct - C - 2A$$

$$At + 2A = Ct$$

$$A(t+2) = Ct$$

$$A = C\left[\frac{t}{t+2}\right]$$

$G(x,t)$ becomes

$$G(x,t) = \begin{cases} C\left[\frac{t}{t+2}\right]x - C & ; 0 \le x \le t \\ Cx - C - 2\left[\frac{t}{t+2}\right]C & ; t < x \le 1 \end{cases}$$

(v) $G'(x,t)$ is Discountinous at $x = t$

Moreover satisfies the discontiniuty condition

$$\lim_{x \to t^-} G'(x,t) - \lim_{x \to t^+} G'(x,t) = \frac{1}{p(t)}$$

$$C\left[\frac{t}{t+2}\right] - C = 1$$

$$Ct - Ct - 2C = (t+2)$$

$$C = -\frac{(t+2)}{2}$$

Finally $G(x,t)$ becomes

$$G(x,t) = \begin{cases} -\frac{(t+2)}{2}\left[\frac{t}{t+2}\right]x + \left(\frac{t+2}{2}\right) & 0 \le x \le t \\ -\frac{(t+2)}{2}x + \left(\frac{t+2}{2}\right) - \frac{2t}{t+2}\left[-\frac{(t+2)}{2} - \right. & t < x \le 1 \end{cases}$$

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$$G(x,t) = \begin{cases} -\frac{t}{2}x + \frac{t}{2} + 1 & ; \ 0 \le x \le t \\ -\left[\frac{t+2}{2}\right]x + \frac{t}{2} + 1 + t & ; \ t < x \le 1 \end{cases}$$

$$G(x,t) = \begin{cases} -\frac{t}{2}x + \frac{t}{2} + 1 & ; \ 0 \le x \le t \\ -\left[\frac{t+2}{2}\right]x + \left(\frac{3t}{2} + 1\right) & ; \ t < x \le 1 \end{cases}$$

A/2016, A/2014, A/2002

Question 7 :- Construct the green's function associated with boundary value problems

$$\frac{d}{dx}\left\{(1-x^2)\frac{du}{dx}\right\} - \frac{h^2}{1-x^2}u + \lambda\gamma(x)u = 0$$

$u(\pm 1)$ are finite

Solution :- The given D-E is

$$[(1-x^2)u']' - \frac{h^2}{1-x^2}u + \lambda\gamma(x)u = 0 \quad \text{---(1)}$$

Comparing with S-L equation

Here $P(x) = 1 - x^2$

i) To construct the green's function we first check if $\lambda = 0$ is an eigen value

put $\lambda = 0$ in equation (1)

$$[(1-x^2)u']' - \frac{h^2}{1-x^2}u = 0$$

put $t = \ln\left[\frac{1+x}{1-x}\right]$

$$t = \ln(1+x) - \ln(1-x)$$

$$\frac{dt}{dx} = \frac{1}{1+x} + \frac{1}{1-x}$$

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$$\frac{dt}{dx} = \frac{2}{1-x^2}$$

$$\text{As } \frac{du}{dx} = \frac{du}{dt}\cdot\frac{dt}{dx} = \frac{2}{1-x^2}\cdot\frac{du}{dt}$$

$$\frac{d^2u}{dx^2} = \frac{d^2u}{dt^2}\cdot\frac{dt}{dx}\cdot\frac{dt}{dx} + \frac{du}{dt}\cdot\frac{d^2t}{dx^2}$$

$$= \frac{d^2u}{dt^2}\left[\frac{dt}{dx}\right]^2 + \frac{du}{dt}\cdot\frac{d^2t}{dx^2}$$

$$= \left[\frac{2}{(1-x^2)}\right]^2\cdot\frac{d^2u}{dt^2} + \frac{4x}{(1-x^2)^2}\cdot\frac{du}{dt}$$

$$\frac{d^2u}{dx^2} = \frac{4}{(1-x^2)^2}u'' + \frac{4x}{(1-x^2)^2}u'$$

we have $(1-x^2)u'' - 2xu' - \frac{h^2}{1-x^2}u = 0$

$$(1-x^2)\left[\frac{4}{(1-x^2)^2}u'' + \frac{4x}{(1-x^2)^2}u'\right] - 2x\left[\frac{2}{(1-x^2)}\frac{du}{dt}\right] - \frac{h^2}{(1-x^2)}u = 0$$

$$\frac{4u''}{(1-x^2)} + \frac{4x}{(1-x^2)}u' - \frac{4x}{(1-x^2)}u' - \frac{h^2}{1-x^2}u = 0$$

$$\frac{4}{(1-x^2)}u'' - \frac{h^2}{1-x^2}u = 0$$

$$\frac{4}{(1-x^2)}\cdot\frac{d^2u}{dt^2} - \frac{h^2}{1-x^2}u = 0$$

Multiply by $(1-x^2)$ on both side

$$4\cdot\frac{d^2u}{dt^2} - h^2u = 0$$

$$4D_t^2u - h^2u = 0$$

$$(4D_t^2 - h^2)u = 0$$

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$$4D_t^2 - h^2 = 0$$

$$D_t = \pm \frac{h}{2}$$

So

$$u = Ae^{\frac{h}{2}t} + Be^{-\frac{h}{2}t}$$

$$u = A(e^t)^{h/2} + B(e^t)^{-h/2}$$

$$u = A\left[e^{\ln\left\{\frac{1+x}{1-x}\right\}}\right]^{\frac{h}{2}} + B\left[e^{\ln\left\{\frac{1+x}{1-x}\right\}}\right]^{-\frac{h}{2}}$$

$$= A\left[\frac{1+x}{1-x}\right]^{\frac{h}{2}} + B\left[\frac{1+x}{1-x}\right]^{-\frac{h}{2}}$$

$$!(x) = A\left[\frac{1+x}{1-x}\right]^{\frac{h}{2}} + B\left[\frac{1-x}{1+x}\right]^{\frac{h}{2}}$$

$u(1)$ is finite

$$u(1) = A\left[\frac{2}{0}\right]^{\frac{h}{2}} + B\left[\frac{0}{2}\right]^{\frac{h}{2}} \text{ is finite}$$

which is only possible if $A = 0$

Also $u(-1)$ is finite

$$u(-1) = A\left[\frac{0}{2}\right]^{\frac{h}{2}} + B\left[\frac{2}{0}\right]^{\frac{h}{2}} \text{ is finite}$$

which is only possible if $B = 0$

For $\lambda = 0$ is not eigen value of given system.

(ii)

Now we Construct the Green's function Associated with B.V.P

$$(1-x^2)G''(x,t) - 2xG'(x,t) - \frac{h^2}{(1-x^2)}G(x,t) = 0$$

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$$G(x,t)=\begin{cases} A\left[\frac{1+x}{1-x}\right]^{\frac{h}{2}} + B\left[\frac{1-x}{1+x}\right]^{\frac{h}{2}} ; & -1\le x\le t \\ C\left[\frac{1+x}{1-x}\right]^{\frac{h}{2}} + D\left[\frac{1-x}{1+x}\right]^{\frac{h}{2}} ; & t\le x\le 1 \end{cases}$$

(iii) $G_1(x,t)$ satisfies the B.C's

$$G_1(1,t) = \text{finite}$$

$$C\left[\frac{2}{0}\right]^{\frac{h}{2}} + D\left[\frac{0}{2}\right]^{\frac{h}{2}} = \text{finite}$$

which is only possible if $C = 0$

Now $G_1(-1,t) = \text{finite}$

$$A\left[\frac{0}{2}\right]^{\frac{h}{2}} + B\left[\frac{2}{0}\right]^{\frac{h}{2}} = \text{finite}$$

which is only possible if $B = 0$

$G_1(x,t)$ becomes

$$G_1(x,t)=\begin{cases} A\left[\frac{1+x}{1-x}\right]^{\frac{h}{2}} ; & -1\le x\le t \\ D\left[\frac{1-x}{1+x}\right]^{\frac{h}{2}} ; & t\le x\le 1 \end{cases}$$

(iv) $G_1(x,t)$ regarded as function of x is continuous at every where at $[-1,1]$, particularly at $x=t$

$$A\left[\frac{1+t}{1-t}\right]^{\frac{h}{2}} = D\left[\frac{1-t}{1+t}\right]^{\frac{h}{2}} = \mathbf{R}\ (\text{say})$$

$$A = R\left[\frac{1-t}{1+t}\right]^{\frac{h}{2}} \text{ and } D = R\left[\frac{1+t}{1-t}\right]^{\frac{h}{2}}$$

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$$\widetilde{G}(x,t) = \begin{cases} R\left[\dfrac{1-t}{1+t}\right]^{\frac{h}{2}}\left[\dfrac{1+x}{1-x}\right]^{\frac{h}{2}} ; & -1 \le x \le t \\ R\left[\dfrac{1+t}{1-t}\right]^{\frac{h}{2}}\left[\dfrac{1-x}{1+x}\right]^{\frac{h}{2}} ; & t < x \le 1 \end{cases}$$

v) $G'(x,t)$ is Discontinuous at $x=t$

Moreover satisfies the Discontinuity Condition

$$\lim_{x\to t^+} G'(x,t) - \lim_{x\to t^-} G'(x,t) = \frac{-1}{\rho(t)}$$

$$\frac{\partial}{\partial t}\left[\frac{1+t}{1-t}\right]^{\frac{h}{2}}\left[\frac{1-t}{1+t}\right]^{\frac{h}{2}} = \frac{h}{2}\left\{\frac{1-t}{1+t}\right\}^{\frac{h}{2}-1}\left[\frac{(1+t)(-1)-(1-t)}{(1+t^2)}\right]$$

$$R\left[\frac{1-t}{1+t}\right]^{\frac{h}{2}}\left[\frac{h}{2}\left\{\frac{1+t}{1-t}\right\}^{\frac{h}{2}-1}\frac{(1-t)-(1+t)(-1)}{(1-t)^2}\right] = \frac{-1}{1-t^2}$$

$$\frac{\partial}{\partial t}\left[\frac{1+t}{1-t}\right]^{\frac{h}{2}}\left[\frac{h}{2}\left\{\frac{1-t}{1+t}\right\}^{\frac{h}{2}-1}\frac{-1-t-1+t}{(1+t)^2}\right]$$

$$R\left[\frac{1-t}{1+t}\right]^{\frac{h}{2}}\left[\frac{h}{2}\left\{\frac{1+t}{1-t}\right\}^{\frac{h}{2}-1}\frac{1-t+1+t}{(1+t)^2}\right] = \frac{-1}{1-t^2}$$

$$\frac{\partial}{\partial t}\left[\frac{1+t}{1-t}\right]^{\frac{h}{2}}\left[\frac{h}{2}\left\{\frac{1-t}{1+t}\right\}^{\frac{h}{2}-1}\frac{-2}{(1+t)^2}\right]$$

$$R\left[\frac{1-t}{1+t}\right]^{\frac{h}{2}}\left[\frac{h}{2}\left\{\frac{1+t}{1-t}\right\}^{\frac{h}{2}-1}\frac{2}{(1-t)^2}\right] = \frac{-1}{1-t^2}$$

$$\frac{\partial}{\partial t}\left[\frac{1-t}{1+t}\right]^{-\frac{h}{2}}\left[\left\{\frac{1-t}{1+t}\right\}^{\frac{h}{2}-1}\cdot\frac{-h}{(1+t)^2}\right]$$

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$$-R\left[\frac{1+t}{1-t}\right]^{-\frac{h}{2}}\left\{\frac{1+t}{1-t}\right\}^{\frac{h}{2}-1}\frac{h}{(1-t)^2}=\frac{-1}{1-t^2}$$

$$R\left[\frac{1-t}{1+t}\right]^{-\frac{h}{2}+\frac{h}{2}-1}\frac{-h}{(1+t)^2}-R\left[\frac{1+t}{1-t}\right]^{-\frac{h}{2}+\frac{h}{2}-1}\frac{h}{(1-t)^2}=\frac{-1}{1-t^2}$$

$$-R\left[\frac{1-t}{1+t}\right]^{-1}\frac{h}{(1+t)^2}-R\left[\frac{1+t}{1-t}\right]^{-1}\frac{h}{(1-t)^2}=\frac{-1}{1-t^2}$$

$$-R\left[\frac{(1+t)}{(1-t)}\right]\frac{h}{(1+t)^2}-R\left[\frac{(1-t)}{(1+t)}\right]\frac{h}{(1-t)^2}=\frac{-1}{1-t^2}$$

$$-\frac{Rh}{1-t^2}-\frac{Rh}{1-t^2}=\frac{-1}{1-t^2}$$

$$\frac{-1}{1-t^2}\left[Rh+Rh\right]=\frac{-1}{1-t^2}$$

$$2Rh=1 \Rightarrow R=\frac{1}{2h}$$

Hence

$$\Gamma(x,t)=\begin{cases}\dfrac{1}{2h}\left[\dfrac{1-t}{1+t}\right]^{\frac{h}{2}}\left[\dfrac{1+x}{1-x}\right]^{\frac{h}{2}}; & -1\le x\le t\\[2ex] \dfrac{1}{2h}\left[\dfrac{1+t}{1-t}\right]^{\frac{h}{2}}\left[\dfrac{1-x}{1+x}\right]^{\frac{h}{2}}; & t<x\le 1\end{cases}$$

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MODIFIED GREEN'S FUNCTION

1/2015, A/2013, A/2012, S/2010, A/2006, A/1999

Question :- Explain the properties of Modified Green's function.

Solution :-

If $\lambda = 0$ is an eigen value of S-L - System

$$L[u] + \lambda \gamma(x) u = 0 \quad , B_1(u) = 0 \;\&\; B_2(u) = 0$$

Then the associated Green's function called modified Green's function.

It is denoted by $G_m(x,t)$ and can be constructed as follows.

1) Let $U_0(x)$ be the normalized eigen function corresponding to $\lambda = 0$

i-e

$$\langle U_0(x), U_0(x) \rangle = \int_a^b U_0(x)\, U_0(x)\, dx = 1$$

2) The modified Green's function $G_m(x,t)$ satisfies the equation

$$L[G_m(x,t)] = U_0(x) \cdot U_0(t)$$

in each sub interval $[a,t]$ and $]t,b]$.

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(3) $G_m(x,t)$ satisfies the Boundary Condition
$B_1[G_m(x,t)] = 0$ and $B_2[G_m(x,t)] = 0$

(4) $G_m(x,t)$ is continous everywhere in the interval $[a,b]$ and in particular at $x = t$

(5) $G_m(x,t)$ satisfies the discontinuity condition
$$\lim_{x \to t^-} G_m'(x,t) - \lim_{x \to t^+} G_m'(x,t) = \frac{1}{p(t)}$$

(6) $G_m(x,t)$ satisfies the orthogonality Condition
$$\int_a^b G_m(x,t)\, u_0(x)\, dx = 0$$

S/2009, A/2007, A/2004, A/1993

Question 1:- Find the Green's function associated with Periodic S-L - System

$u'' + \lambda = 0 \quad ; \quad u(0) = u(1)$

$u'(0) = u'(1)$

Solution:- The given D-E is

$$u'' + \lambda u = 0 \quad \text{——(1)}$$

Re-write the equation (1) in S-L - form

$$(u')' + \lambda u = 0$$

Here $p(x) = 1 \neq 0$

Take $\lambda = 0$ in equation (1)

$$u'' = 0$$

integrating both sides

$$u' = A$$

again integrating both side $u = Ax + B$

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Apply the boundary conditions.

i-e $u(0) = u(1)$

$B = A + B \Rightarrow A = 0$

also $u'(0) = u'(1)$

$\Rightarrow A = A$

Second condition does not give any infomation about the constant B

$\Rightarrow u(x) = B$

So for $\lambda = 0$ the solution is non-trivial

Therefore $\lambda = 0$ is an eigen value of the given system.

So we can construct the modified Green's function such that

i) Let $U_0(x) = B$ be the normalized eigen function corresponding to $\lambda = 0$

i-e $\int_0^1 U_0(x)\, U_0(x)\, dx = 1$

$\int_0^1 (B)(B)\, dx = 1$

$B^2 \int_0^1 dx = 1$

$B^2 [x]_0^1 = 1$

$B^2 = 1 \Rightarrow B = 1$

Hence $\Rightarrow u(x) = 1$

ii) The modified Green's function satisfies the equation

$G_m''(x,t) = U_0(x) \cdot \dot{U}_0(t)$

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$$G_m''(x,t) = 1$$

$$G_m'(x,t) = x + A$$

$$G_m(x,t) = \frac{x^2}{2} + Ax + B$$

Hence for the modified Green's function has the form

$$G_m(x,t) = \begin{cases} \frac{x^2}{2} + Ax + B & ; \ 0 \le x \le t \\ \frac{x^2}{2} + Cx + D & ; \ t < x \le 1 \end{cases}$$

(iii) $G_m(x,t)$ satisfies the boundary conditions

$$G_m(0,t) = G_m(1,t)$$

$$B = \frac{1}{2} + C + D$$

also $$G_m'(x,t) = \begin{cases} x + A & ; \ 0 \le x \le t \\ x + C & ; \ t < x \le 1 \end{cases}$$

$$G_m'(0,t) = G_m'(1,t)$$

$$0 + A = 1 + C \Rightarrow A = 1 + C$$

$G_m(x,t)$ becomes

$$G_m(x,t) = \begin{cases} \frac{x^2}{2} + (1+C)x + \frac{1}{2} + C + D & ; \ 0 \le x \le t \\ \frac{x^2}{2} + Cx + D & ; \ t \le x \le 1 \end{cases}$$

$$G_m(x,t) = \begin{cases} \frac{x^2}{2} + x + Cx + \frac{1}{2} + C + D & ; \ 0 \le x \le t \\ \frac{x^2}{2} + Cx + D & ; \ t < x \le 1 \end{cases}$$

$$G_m(x,t) = \begin{cases} \frac{x^2}{2} + x + (x+1)C + \frac{1}{2} + D & ; \ 0 \le x \le t \\ \frac{x^2}{2} + Cx + D & ; \ t < x \le 1 \end{cases}$$

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iv) $G_m(x,t)$ regarding as the function of x is continuous at everywhere in $[a,1]$ and in particular at $x=t$

$$\frac{t^2}{2}+t+(t+1)C+\frac{1}{2}+D=\frac{t^2}{2}+Ct+D$$

$$t+tC+C+\frac{1}{2}=Ct$$

$$C=-t-\frac{1}{2}$$

$$C=-\left[\frac{2t+1}{2}\right]$$

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$$G_m(x,t)=\begin{cases}\frac{x^2}{2}+x+(x+1)\left(\frac{-2t-1}{2}\right)+\frac{1}{2}+D & ;\ 0\le x\le t\\ \frac{x^2}{2}-\left(\frac{2t+1}{2}\right)x+D & ;\ t<x\le 1\end{cases}$$

$$\hat{G}_m(x,t)=\begin{cases}\frac{x^2}{2}+x-\left[\frac{2t+1}{2}\right]x-\left[\frac{2t+1}{2}\right]+\frac{1}{2}+D & ;\ 0\le x\le t\\ \frac{x^2}{2}-\left[\frac{2t+1}{2}\right]x+D & ;\ t<x\le 1\end{cases}$$

$$;(x,t)=\begin{cases}\frac{x^2}{2}+\left[1-\frac{2t+1}{2}\right]x-\left[\frac{2t+1}{2}\right]+\frac{1}{2}+D & ;\ 0\le x\le t\\ \frac{x^2}{2}-\left[\frac{2t+1}{2}\right]x+D & ,\ t<x\le 1\end{cases}$$

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') $G_m'(x,t)$ is discontinuous at $x=t$ Moreover satisfies the discontinuty dition

$$\lim_{x\to t^-} G_m'(x,t)-\lim_{x\to t^+} G_m'(x,t)=\frac{1}{\rho(t)}$$

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$$t + \left[1 - \frac{2t+1}{2}\right] - \left[t - \left(\frac{2t+1}{2}\right)\right] = 1$$

$$t + 1 - t - \frac{1}{2} - t + t + \frac{1}{2} = 1 \Rightarrow 1 = 1$$

which gives no information about the constant D.

(vi) $G_m(x,t)$ satisfies the orthogonality condition.

$$\therefore \int_0^1 G_m(x,t) U_0(x)\, dx = 0$$

$$\int_0^t \left[\frac{x^2}{2} + \left[\frac{1}{2} - t\right]x - t + D\right) dx + \int_t^1 \left(\frac{x^2}{2} - \left[t + \frac{1}{2}\right]x + D\right) dx =$$

$$\left|\frac{x^3}{6} + \left(\frac{1}{2} - t\right)\frac{x^2}{2} - tx + Dx\right|_0^t + \left|\frac{x^3}{6} - \left(t + \frac{1}{2}\right)\frac{x^2}{2} + Dx\right|_t^1 = 0$$

$$\frac{t^3}{6} + \left(\frac{1}{2} - t\right)\frac{t^2}{2} - t^2 + Dt + \frac{1}{6} - \left(t + \frac{1}{2}\right)\frac{1}{2} + D - \frac{t^3}{6}$$

$$+ \left(t + \frac{1}{2}\right)\frac{t^2}{2} - Dt = 0$$

$$\frac{t^2}{4} - \frac{t^3}{2} - t^2 + \frac{1}{6} - \frac{t}{2} - \frac{1}{4} + D + \frac{t^3}{2} + \frac{t^2}{4} = 0$$

$$\frac{t^2}{4} + \frac{t^2}{4} - t^2 - \frac{t}{2} + \frac{1}{6} - \frac{1}{4} + D = 0$$

$$-\frac{t^2}{2} - \frac{t}{2} - \frac{1}{12} + D = 0$$

$$D = \frac{t^2}{2} + \frac{t}{2} + \frac{1}{12}$$

$G_m(x,t)$ becomes

$$G_m(x,t) = \begin{cases} \frac{x^2}{2} + \left(1 - \frac{2t+1}{2}\right)x - \left(\frac{2t+1}{2}\right) + \frac{1}{2} + \frac{t^2}{2} + \frac{t}{2} + \frac{1}{12} \\ \qquad ; 0 \le x \le t \\ \frac{x^2}{2} - \left(\frac{2t+1}{2}\right)x + \frac{t^2}{2} + \frac{t}{2} + \frac{1}{12} \quad ; t < x \le 1 \end{cases}$$

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$$m(x,t) = \begin{cases} \frac{x^2}{2} - \left[\frac{1}{2} - t\right]x - t + \frac{t^2}{2} + \frac{t}{2} + \frac{1}{12} & ; 0 \le x \le t \\ \frac{x^2}{2} - \left[\frac{1}{2} + t\right]x + \frac{t^2}{2} + \frac{t}{2} + \frac{1}{12} & ; t < x \le 1 \end{cases}$$

A/2008, A/2006

Question 2:- Construct the Green's function associated with the system.

$$u'' + \lambda r(x) u = 0$$

$$u'(0) = 0 \quad , \quad u'(1) = 0$$

Solution:- The given differential equation is

$$u'' + \lambda r(x) u = 0 \quad \text{—(i)}$$

Re-write equation (i) in S-L form

$$(1 \cdot u')' + \lambda r(x) u = 0$$

Here $p(x) = 1 \neq 0$

put $\lambda = 0$ in equation (i)

$$u'' = 0$$

integrating both side $u' = A$

again integrating $u = Ax + B$

Now apply the boundary conditions

$u'(0) = 0 \Rightarrow A = 0$

$u'(1) = 0 \Rightarrow A = 0$

$\Rightarrow u(x) = B$

There is no information about the constant B.

Therefor for $\lambda = 0$ the solution is non-trivial

So $\lambda = 0$ is an eigen value of the given system. Now we construct the modified Green's function as follows

(i) Let $U_0(x) = B$ be the normalized

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eigen function, corresponding to $\lambda = 0$

i.e $\int_0^1 U_0(x) U_0(x)\, dx = 1$

$$\int_0^1 B^2 \cdot dx = 1$$

$$B^2 [x]_0^1 = 1$$

$$B^2 [1 - 0] = 1 \Rightarrow B^2 = 1$$

$$\Rightarrow B = 1$$

$$U(x) = 1$$

∴ Normalizer will give any positive value

$B = \pm 1$,

$B = 1$ ✓

(ii) The modified Green's function satisfies the equation in each subinterval $[0,t]$ & $(t,1]$.

$$G_m''(x,t) = U_0(x) \cdot U_0(t)$$

$$G_m''(x,t) = 1$$

integrating $G_m'(x,t) = x + A$

$$G_m(x,t) = \frac{x^2}{2} + Ax + B$$

These for we can write

$$G_m(x,t) = \begin{cases} \frac{x^2}{2} + Ax + B & ; \ 0 \le x \le t \\ \frac{x^2}{2} + Cx + D & ; \ t < x \le 1 \end{cases}$$

(iii) $G_m(x,t)$ satisfies the boundary condition

$$G_m'(0,t) \Rightarrow 0 + A = 0 \Rightarrow A = 0$$

$$G_m'(1,t) \Rightarrow 1 + C = 0 \Rightarrow C = -1$$

$$G_m(x,t) = \begin{cases} \frac{x^2}{2} + B & ; \ 0 \le x \le t \\ \frac{x^2}{2} - x + D & ; \ t < x \le 1 \end{cases}$$

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iv) $G_m(x,t)$ regarding as function of x is continuous at everywhere in $[0,1]$ and in particular at $x=t$

$$\frac{t^2}{2} + B = \frac{t^2}{2} - t + D$$

$$B = D - t$$

$$G_m(x,t) = \begin{cases} \frac{x^2}{2} - t + D & ; \ 0 \le x \le t \\ \frac{x^2}{2} - x + D & ; \ t < x \le 1 \end{cases}$$

v) $G_m'(x,t)$ is discontinuous at $x=t$ Moreover satisfies the discontinuity condition.

$$\lim_{x \to t^+} G_m'(x,t) - \lim_{x \to t^-} G_m'(x,t) = \frac{-1}{\rho(t)}$$

$$t - t - 1 = -1$$

$$\Rightarrow \ 1 = 1$$

Discontinuity condition does not give any information about the constant D.

vi) $G_m(x,t)$ satisfies the orthogonality conditions.

$$\int_a^b G_m(x,t)\, u_0(x)\, dx = 0$$

$$\int_0^t \left(\frac{x^2}{2} - t + D\right) dx + \int_t^1 \left(\frac{x^2}{2} - x + D\right) dx = 0$$

$$\left[\frac{x^3}{6} - tx + Dx\right]_0^t + \left[\frac{x^3}{6} - \frac{x^2}{2} + Dx\right]_t^1 = 0$$

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$$\frac{t^3}{6} - t^2 + Dt + \frac{1}{6} - \frac{1}{2} + D - \frac{t^3}{6} + \frac{t^2}{2} - Dt = 0$$

$$-t^2 + \frac{t^2}{2} + \frac{1}{6} - \frac{1}{2} + D = 0$$

$$-\frac{t^2}{2} - \frac{1}{3} + D = 0 \Rightarrow D = \frac{t^2}{2} + \frac{1}{3}$$

Hence $G_m(x,t)$ becomes

$$G_m'(x,t) = \begin{cases} \frac{x^2}{2} - t + \frac{t^2}{2} + \frac{1}{3} & ; \quad 0 \le x \le t \\ \frac{x^2}{2} - x + \frac{t^2}{2} + \frac{1}{3} & ; \quad t < x \le 1 \end{cases}$$

A/2005, A/1991

Question 3:- Find Green's function for problem

$$u'' + \lambda u = 0$$

$$u(-1) = u(1), \quad u'(-1) = u'(1)$$

Solution:- The given D-E is

$$u'' + \lambda u = 0 \quad \text{---(i)}$$

Re-write equation (i) in S-L form

$$(1 \cdot u')' + \lambda u = 0$$

Here $p(x) = 1 \neq 0$

put $\lambda = 0$ in equation (i)

$$u'' = 0$$

integrating both side $u' = A$

again integrating both sides $u = Ax + B$

Apply the boundary conditions

$$u(-1) = u(1)$$

$$-A + B = A + B$$

$$-A = A$$

$$2A = 0$$

$$A = 0$$

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$u'(-1) = u'(1) \Rightarrow A = A$

$\Rightarrow u(x) = B$

There is no information about the constant B
There for $\lambda = 0$ the solution is non-trivial
There for $\lambda = 0$ is an eigen value of the given system. Now we construct the modified Green's function such that

i) Let $U_0(x) = B$ is the normalized eigen function corresponding to $\lambda = 0$ i.e

$$\int_{-1}^{1} U_0(x) \cdot U_0(x)\, dx = 1$$

$$\int_{-1}^{1} B^2 \cdot dx = 1$$

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$$B^2 [x]_{-1}^{1} = 1$$

$$B^2[1 - (-1)] = 1 \Rightarrow 2B^2 = 1$$

$$B^2 = \frac{1}{2}$$

$$B = \frac{1}{\sqrt{2}}$$

Then $U_0(x) = \frac{1}{\sqrt{2}}$ is normalized eigen function and $U_0(t) = \frac{1}{\sqrt{2}}$

ii) $G_m''(x,t)$ satisfies the D-E

$$G_m''(x,t) = U_0(x) \cdot U_0(t)$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}$$

$$G_m''(x,t) = \frac{1}{2}$$

integrating both side

$$G_m'(x,t) = \frac{x}{2} + A$$

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Again integrating both side

$$G_m(x,t) = \frac{x^2}{4} + Ax + B$$

Therefor the modified Green's function has the foom

$$G_m(x,t) = \begin{cases} \frac{x^2}{4} + Ax + B \; ; & -1 \leq x \leq t \\ \frac{x^2}{4} + Cx + D \; ; & t < x \leq 1 \end{cases}$$

(iii) $G_m(x,t)$ satisfies the B.Cs

$$G_m(-1,t) = G_m(1,t)$$

$$\frac{1}{4} - A + B = \frac{1}{4} + C + D$$

$$D = B - A - C \quad \text{——} (a)$$

And $G_m'(-1,t) = G_m'(1,t)$

$$-\frac{1}{2} + A = \frac{1}{2} + C$$

$$-\frac{1}{2} - \frac{1}{2} + A = C \Rightarrow \boxed{C = A - 1}$$

put the value of C in above equation

$$D = B - A - A + 1$$

$$D = B - 2A + 1$$

So $G_m(x,t)$ becomes

$$G_m(x,t) = \begin{cases} \frac{x^2}{4} + Ax + B \; ; & -1 \leq x \leq t \\ \frac{x^2}{4} + (A-1)x + B - 2A + 1 \; ; & t < x \leq 1 \end{cases}$$

(iv) $G_m(x,t)$ is continous at all points in each interval $[-1,1]$ and in particular at $x = t$

$$\frac{t^2}{4} + At + B = \frac{t^2}{4} + (A-1)t + B - 2A + 1$$

$$0 = -t - 2A + 1$$

$$2A = 1 - t \Rightarrow A = \frac{1-t}{2}$$

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$$G_m(x,t) = \begin{cases} \frac{x^2}{4} + \frac{x(1-t)}{2} + B & ; \quad -1 \le x \le t \\ \frac{x^2}{4} - \frac{x(1+t)}{2} + B + t & ; \quad t < x \le 1 \end{cases}$$

v) $G'(x,t)$ is discontinous at $x=t$ Moreover satisfies the discontinuity Conditions.

$$\lim_{x \to t^+} G'_m(x,t) - \lim_{x \to t^-} G'_m(x,t) = \frac{-1}{P(t)}$$

$$\frac{t}{2} - \left[\frac{1}{2} + \frac{t}{2}\right] - \frac{t}{2} - \left[\frac{1}{2} - \frac{t}{2}\right] = -1$$

$$-\frac{1}{2} - \frac{t}{2} - \frac{1}{2} + \frac{t}{2} = -1$$

$$-1 = -1$$

The discountinuity condition does not help in finding the value of B.

vi) $G_m(x,t)$ satisfies the orthogonality condition

$$\int_{-1}^{1} G_m(x,t)\, U_0(x) \cdot dx = 1$$

$$\int_{-1}^{t} G_m(x,t)\, U_0(x)\, dx + \int_{t}^{1} G_m(x,t)\, U_0(x)\, dx = 0$$

$$\int_{-1}^{t} \left(\frac{x^2}{4} + \left\{\frac{1}{2} - \frac{t}{2}\right\}x + B\right) dx + \int_{t}^{1} \left(\frac{x^2}{4} - \left\{\frac{1+t}{2}\right\}x + B + t\right) dx = 0$$

$$\frac{x^3}{12} + \left[\frac{1-t}{2}\right]\frac{x^2}{2} + Bx \Big|_{-1}^{t} + \left[\frac{x^3}{12} - \left\{\frac{1+t}{2}\right\}\frac{x^2}{2} + Bx + tx\right]_{t}^{1} = 0$$

$$\frac{t^3}{12} + \left[\frac{1-t}{2}\right]\frac{t^2}{2} + Bt\Big] - \left[-\frac{1}{12} + \left(\frac{1-t}{2}\right)\frac{1}{2} - B\right]$$

$$+\left[\frac{1}{12} - \left(\frac{1+t}{2}\right)\frac{1}{2} + B + t\right] - \left[\frac{t^3}{12} - \left(\frac{1+t}{2}\right)\frac{t^2}{2} + Bt + t^2\right] = 0$$

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$$\frac{t^3}{12}+\frac{t^2}{4}-\frac{t^3}{4}+Bt+\frac{1}{12}-\frac{1}{4}+\frac{t}{4}+B+\frac{1}{12}-\frac{1}{4}$$

$$-\frac{t}{4}+B+t-\frac{t^3}{12}+\frac{t^2}{4}+\frac{t^3}{4}-Bt-t^2=0$$

$$2B-\frac{t^2}{2}+t-\frac{1}{3}=0 \Rightarrow B=\frac{t^2}{4}-\frac{t}{2}+\frac{1}{6}$$

$$B=\frac{3t^2-6t+2}{12}$$

Hence

$$G_m(x,t)=\begin{cases}\frac{x^2}{4}+\frac{t^2}{4}-\frac{xt}{2}+\frac{x}{2}-\frac{t}{2}+\frac{1}{6} & ; \quad -1\leq x\leq t\\ \frac{x^2}{4}+\frac{t^2}{4}-\frac{xt}{2}-\frac{x}{2}-\frac{t}{2}+\frac{1}{6} & ; \quad t<x\leq 1\end{cases}$$

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9/2015

Question :- Compute the Green's function associated with boundary value problem

$$y''(x) = f(x)$$

$$y(0) = 0 \quad ; \quad y'(1) = 0$$

where $f(x)$ is well-behaved arbitrary function.

Solution :-

As $y(x) = \int G(x,t)\, f(t)\, dt$

where the $G(x,t)$ ° green's function given

$$G(x,t) = \sum_{n}' \frac{y_n(x)\, y_n(t)}{\lambda - \lambda_n} \quad \text{---- (i)}$$

In the given problem $\lambda = 0$

λ_n and y_n are eigen value problem

$$L\, y_n = \lambda_n y_n \qquad y(0) = 0 \,, \; y'(1) = 0$$

$$\frac{d^2 y_n}{dx^2} - \lambda y = 0$$

Auxillary equation is

$$D^2 - \lambda y = 0$$

$$D = \sqrt{\lambda}$$

$$y = c_1 e^{\sqrt{\lambda}\, x} + c_2 e^{-\sqrt{\lambda}\, x}$$

$$y(0) = 0 \Rightarrow c_1 + c_2 = 0 \text{ or } c_2 = -c_1$$

$$y'(1) = \sqrt{\lambda}\left[c_1 e^{\sqrt{\lambda}\, x} - c_2 e^{-\sqrt{\lambda}\, x}\right]$$

$$y'(1) = 0 \Rightarrow c_1 e^{\sqrt{\lambda}} - c_2 e^{-\sqrt{\lambda}} = 0$$

$$c_1 e^{\sqrt{\lambda}} + c_1 e^{-\sqrt{\lambda}} = 0$$

$$c_1\left[e^{\sqrt{\lambda}} + e^{-\sqrt{\lambda}}\right] = 0$$

$$c_1 \neq 0 \qquad e^{\sqrt{\lambda}} + e^{-\sqrt{\lambda}} = 0$$

$$e^{\sqrt{\lambda}} = \frac{-1}{e^{\sqrt{\lambda}}}$$

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$$e^{2\sqrt{\lambda}} = \frac{1}{e^{\pi i}} = e^{\pi i + 2n\pi i}$$

$$e^{2\sqrt{\lambda}} = e^{\pi i (2n+1)}$$

$$2\sqrt{\lambda} = \pi i (2n+1)$$

$$\sqrt{\lambda} = \frac{\pi i (2n+1)}{2}$$

$$\lambda_n = \frac{-\pi^2 (2n+1)^2}{4} \qquad n = 0, 1, 2, 3 \text{---}$$

The corresponding eigen vectors are

$$y_n = c_{1n} e^{\sqrt{\lambda} nx} + c_{2n} e^{-\sqrt{\lambda} nx}$$

$$= c_{1n} e^{\sqrt{\lambda} nx} - c_n e^{-\sqrt{\lambda} nx}$$

$$= c_{1n} \left[e^{\sqrt{\lambda} nx} - e^{-\sqrt{\lambda} nx} \right]$$

$$= c_{1n} \left[e^{\pi i \left[\frac{2n+1}{2}\right] x} - e^{-\pi i \left[\frac{2n+1}{2}\right] x} \right]$$

$$y_n = 2i\, c_{1n} \left[\frac{e^{\pi i \left[\frac{2n+1}{2}\right] x} - e^{-\pi i \left[\frac{2n+1}{2}\right] x}}{2i} \right]$$

$$y_n = 2i\, c_{1n} \sin \frac{\pi}{2} (2n+1) x \qquad n = 0, 1, 2, 3 \text{---}$$

$$y_n = A_n \sin \frac{\pi}{2} (2n+1) x \quad ; \quad n = 0, 1, 2, 3 \text{---}$$

$$A_n = 2i\, c_n$$

For Normalized eigen function

$$\int_0^1 y_n^2 \cdot dx = 1$$

$$A_n^2 \int_0^1 \sin^2 \frac{\pi}{2} (2n+1) x \cdot dx = 1$$

$$A_n^2 \int_0^1 \frac{1 - \cos 2\left(\frac{\pi}{2} (2n+1) x\right)}{2} \cdot dx = 1$$

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$$\frac{A_n^2}{2}\int_0^1 dx - \frac{A_n^2}{2}\int_0^1 \sin x\left(\frac{\pi}{x}(2n+1)x\right)dx = 1$$

$$\frac{A_n^2}{2}\left[x\Big|_0^1 - \frac{A_n^2}{2}\left[\frac{\sin(\pi(2n+1)x)}{\pi(2n+1)}\Big|_0^1\right] = 1\right.$$

$$\frac{A_n^2}{2}[1-0] - \frac{A_n^2}{2}[0-0] = 1$$

$$\frac{A_n^2}{2} = 1 \Rightarrow A_n^2 = 2 \Rightarrow A_n = \sqrt{2}$$

Thus $y_n = \sqrt{2}\sin\left(\frac{2n+1}{2}\right)\pi x$

By using formula in π

$$G(x,t) = \sum_n \frac{y_n(x)\,y_n(t)}{\lambda - \lambda_n}$$

$$= \sum_n \frac{\sqrt{2}\sin\left[\frac{2n+1}{2}\right]\pi x \cdot \sqrt{2}\sin\left[\frac{2n+1}{2}\right]\pi t}{\pi^2\left[\frac{2n+1}{2}\right]^2}$$

$$= \frac{2}{\pi^2}\sum_n \frac{\sin\left[\frac{2n+1}{2}\right]\pi x \cdot \sin\left[\frac{2n+1}{2}\right]\pi t}{\frac{(2n+1)^2}{4}}$$

Hence $G_\pi(x,t) = \frac{8}{\pi^2}\sum_n \frac{\sin\left[\frac{2n+1}{2}\right]\pi x \cdot \sin\left[\frac{2n+1}{2}\right]\pi t}{(2n+1)^2}$

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From Sharaqpur Sharif

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BS- Math (8-Smester) A/2015

Question :- Find the Green's function associated with boundary value problem defined by

$$y'' + \frac{1}{4}y = f(x) \quad , y(0) = 0, \; y(\pi) = 0$$

Also solve the problem when $f(x) = \sin 2x$

Solution :-

The associated Eigen value of S-L Problem defined by the given equation is

$$y'' + \frac{1}{4}y = f(x) \qquad Ly = \lambda_n y$$

$$y'' + \frac{1}{4}y = \lambda_n y$$

$$\frac{d^2y}{dx^2} + \frac{y}{4} - \lambda_n y = 0$$

Auxillary Equation is

$$D^2 + \frac{1}{4} - \lambda_n = 0$$

$$D^2 = -\frac{1}{4} + \lambda_n = -\left[\frac{1}{4} - \lambda_n\right]$$

$$D = \pm\sqrt{\frac{1}{4} - \lambda_n}\, i$$

$$y = c_1 \cos\left[\sqrt{\frac{1}{4} - \lambda_n}\right]x + c_2 \sin\left[\sqrt{\frac{1}{4} - \lambda_n}\right]x \quad \text{—— (1)}$$

Applying Boundary conditions in (1)

$$y(0) = 0 \Rightarrow c_1 = 0$$

$$y = c_2 \sin\left[\sqrt{\frac{1}{4} - \lambda_n}\right]x$$

$$y(\pi) = 0$$

$$c_2 \sin\left[\sqrt{\frac{1}{4} - \lambda_n}\right]\pi = 0 \qquad \because c_2 \neq 0$$

$$\sin\left[\sqrt{\frac{1}{4} - \lambda_n}\right]\pi = 0$$

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$$\sqrt{\frac{1}{4}-\lambda_n}\,(\pi) = \sin^{-1}(0)$$

$$\sqrt{\frac{1}{4}-\lambda_n}\,(\pi) = n\pi \qquad n = 1, 2, 3 \ldots$$

$$\sqrt{\frac{1}{4}-\lambda_n} = n$$

$$\Rightarrow n^2 = \frac{1}{4}-\lambda_n$$

$$n^2 - \frac{1}{4} = -\lambda_n \Rightarrow \lambda_n = \frac{1}{4} - n^2$$

Thus $y_n = C_{2n} \sin n\pi$

$$y_n = A_n \sin n\pi \qquad (\because C_{2n} = A_n)$$

To normalize y_n

$$\int_0^{\pi} y_n^2 \, dx = 1$$

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$$\int_0^{\pi} A_n^2 \sin^2 nx \, dx = 1$$

$$A_n^2 \int_0^{\pi} \sin^2 nx \cdot dx = 1$$

$$A_n^2 \int_0^{\pi} \left[\frac{1-\cos 2nx}{2}\right] dx = 1$$

$$\frac{A_n^2}{2} \int_0^{\pi} (1-\cos 2nx)\, dx = 1$$

$$\frac{A_n^2}{2}\left[x\Big|_0^{\pi} - \left[\frac{\sin 2nx}{2n}\Big|_0^{\pi}\right]\right] = 1$$

$$\frac{A_n^2}{2}\left[(\pi - 0) - \left[\frac{\sin 2n\pi}{2n} - 0\right]\right] = 1$$

$$\frac{A_n^2}{2}(\pi) = 1 \Rightarrow A_n^2 = \frac{2}{\pi} \Rightarrow A_n = \sqrt{\frac{2}{\pi}}$$

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Therefor $y_n = \sqrt{\frac{2}{\pi}} \sin nx$

The associated Green's function is given by

$$G(x,t) = \sum_{n=1}^{\infty} \frac{y_n(x)\, y_n(t)}{\lambda - \lambda_n} \quad , \quad \lambda = 0$$

$$G(x,t) = \sum_{n=1}^{\infty} \frac{y_n(x) - y_n(t)}{-\lambda_n}$$

$$G(x,t) = -\sum_{n=1}^{\infty} \sqrt{\frac{2}{\pi}} \sin nx \cdot \sqrt{\frac{2}{\pi}} \sin nt \cdot dt$$

$$\frac{1}{4} - n^2$$

$$G(x,t) = -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin nx \cdot \sin nt}{\frac{1}{4} - n^2}$$

The solution of problem is given by

$$y(x) = \int_0^{\pi} G(x,t) \cdot f(t) \cdot dt$$

$$\Rightarrow \boldsymbol{f(x) = \sin 2x}$$

$$y(x) = \int_0^{\pi} -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin nx \cdot \sin nt}{\frac{1}{4} - n^2} \cdot \sin nt \cdot dt$$

$$= -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin nx}{\frac{1}{4} - n^2} \int_0^{\pi} \frac{2(\sin nt \cdot \sin t)}{2} \cdot dt$$

$$= -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin nx}{\frac{1}{4} - n^2} \int_0^{\pi} \{\cos(2-n)t + \cos(2+n)t\} dt$$

$$y(x) = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin nx}{\frac{1}{4} - n^2} \left[\frac{\sin(2-n)t}{(2-n)} \Big|_0^{\pi} - \frac{\sin(2+n)t}{(2+n)} \Big|_0^{\pi} \right]$$

$$y(x) = 0 \qquad (\text{for } n \neq 2)$$

when $n = 2$ Then

$$y(x) = -\frac{2}{\pi} \cdot \frac{\sin 2x}{\frac{1}{4} - 2^2} \int_0^{\pi} \sin^2 2t \cdot dt$$

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$$y(x) = \frac{-2}{\pi} \frac{\sin 2x}{\left(\frac{-15}{4}\right)} \cdot \int_0^{\pi} \left(\frac{1-\cos 4t}{2}\right) \cdot dt$$

$$y(x) = \frac{4}{15\pi} \sin 2x \left[[t]_0^{\pi} - \left[\frac{\sin 4t}{4} \right]_0^{\pi} \right]$$

$$y(x) = \frac{4}{15\pi} \sin 2x \left[(\pi - 0) - 0 \right]$$

$$y(x) = \frac{4}{15\pi} 15 \sin 2x \ (\pi)$$

$$y(x) = \frac{4}{15} \sin 2x$$

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Question :- $y'' + y = 3 \sin 2\pi x$

$y(0) = 0, \quad y(1) = 0$

Solution :- Given $y'' + y = 3 \sin 2\pi x$

Comparing with S-L equation

$y = \lambda r(x) y \qquad \lambda = 1 \ , \ r = 1$

Thus $L y_n = -\lambda_n y_n$

or $\frac{d^2y}{dx^2} + \lambda_n y = 0$

only if λ = +ve we get non trivial solution $y = C_1 \cos \sqrt{\lambda} x + C_2 \sin \sqrt{\lambda} x$

$y(0) = 0 \Rightarrow C_1 = 0$

$\Rightarrow y = C_2 \sin \sqrt{\lambda} x$

$y(1) = 0 \Rightarrow C_2 \sin \sqrt{\lambda} = 0$

$\sin \sqrt{\lambda} = 0 \qquad (\because C_2 \neq 0)$

$\sqrt{\lambda} = \sin^{-1}(0) = n\pi$

$\lambda = n^2 \pi^2$

Eigen function :-

$y_n = A_n \sin n\pi x \qquad n = 1, 2, 3 \text{---}$

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By Normalizing y_n $\int_0^1 y_n^2 \cdot dx = 1$

$$A_n^2 \int_0^1 \sin^2 n\pi x = 1$$

$$A_n^2 \int_0^1 \left(\frac{1-\cos 2n\pi x}{2}\right) dx = 1$$

$$\frac{A_n^2}{2}\int_0^1 dx - \frac{A_n^2}{2}\int_0^1 \cos 2n\pi x \cdot dx = 1$$

$$\frac{A_n^2}{2}\left[x\right|_0^1 - \frac{A_n^2}{2}\left[\frac{\sin 2n\pi x}{2n\pi}\right|_0^1 = 1$$

$$\frac{A_n^2}{2}[1-0] - \frac{A_n^2}{2}(0-0) = 1$$

$$\frac{A_n^2}{2} = 1 \Rightarrow A_n = \sqrt{2} \qquad n = 1,2,3 -$$

$\therefore y_n = \sqrt{2} \sin \sqrt{\lambda} x$

$= \sqrt{2} \sin n\pi x \qquad n = 1,2,3 \text{ ——}$

As we know that

$y(x) = \int_0^1 G_f(x,t) \cdot f(t) \cdot dt$

where $G_f(x,t)$ is Green's function

and $G_f(x,t) = \sum_n \frac{y_n(x) \cdot y_n(t)}{\lambda - \lambda_n}$

$$G_f(x,t) = \sum_n \frac{\sqrt{2}\sin n\pi x \cdot \sqrt{2}\sin n\pi t}{\lambda - \lambda_n}$$

$$G_f(x,t) = \sum_n \frac{\sqrt{2}\sin n\pi x \cdot \sqrt{2}\sin n\pi t}{1 - n^2\pi^2}$$

Here $f(x) = 3\sin 2\pi x$

As $y(x) = \int_0^1 G_f(x,t) \cdot f(t) \cdot dt$

$$= \int_0^1 2\sum_{n=1}^{\infty} \frac{\sin n\pi x \cdot \sin n\pi t}{1 - n^2\pi^2} (3\sin 2\pi t)\, dt$$

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$$= 3\sum_{n=1}^{\infty} \frac{\sin n\pi x}{1-n^2\pi^2} \int_0^1 2\sin \pi t \cdot \sin 2\pi t \cdot dt$$

$$= 3\sum_{n=1}^{\infty} \frac{\sin n\pi x}{1-n^2\pi^2} \int_0^1 \{\cos(2-n)\pi t - \cos(n+2)\pi t\} dt$$

$$y(x) = 3\sum_{n=1}^{\infty} \frac{\sin n\pi x}{1-n^2\pi^2} \left\{ \frac{\sin(2-n)\pi t}{(2-n)\pi} \Big|_0^1 - \left[\frac{\sin(n+2)\pi t}{(n+2)\pi} \Big|_0^1 \right.\right.$$

$$= 0 \quad \text{for } n \neq 2$$

when $n=2$

$$y(x) = \frac{6}{1-4\pi^2} \int_0^1 \sin 2\pi x \cdot \sin 2\pi t \cdot \sin 2\pi t \, dt$$

$$= \frac{6\sin 2\pi x}{1-4\pi^2} \int_0^1 \sin^2 2\pi t \cdot dt$$

$$= \frac{3\,6\sin 2\pi x}{1-4\pi^2} \int_0^1 \frac{1-\cos 4\pi t}{2} \, dt$$

$$= \frac{3\sin 2\pi x}{1-4\pi^2} \left[[t]_0^1 - \left\{ \frac{\sin 4\pi t}{4\pi} \Big|_0^1 \right] \right.$$

$$= \frac{3\sin 2\pi x}{1-4\pi^2} \left[(1-0) - (0-0) \right]$$

$$y(x) = \frac{3\sin 2\pi x}{1-4\pi^2}$$

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C-R Msc Mathematics (2014-2016)

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VARIATIONAL METHODS

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FUNCTIONAL :- let M be the set of functions defined over the interval [a, b]. If there is a rule which assigns each function of M to a real number "I" then I is called functional from M to R.

$$M \xrightarrow{[a,b]} R\{I\}$$

Stationary values :- The maximum or minimum value of a function or functional is called stationary value.

Extremal :- The curve $y = f(x)$ along which a functional "I" takes the stationary value is called extremal.

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A/2016, A/2015, A/2011 (only statment)

FUNDAMENTAL THEOREM OF CALCULUS OF VARIATION

Statment:- (For one independent variable)

If $f(x)$ is continous in the interval x_1, x_2) and the integral $\int_{x_1}^{x_2} f(x) \cdot g(x) \cdot dx$ is identicaly zero i.e $\int_{x_1}^{x_2} f(x) \cdot g(x) \cdot dx = 0$

where $g(x)$ satisfies the conditions.

(i) It is arbitrary function with continuous derivative in the interval (x_1, x_2).

ii) $g(x_1) = g(x_2) = 0$

Then $f(x) = 0 \quad \forall x \in [x_1, x_2]$

Proof:- We prove by contraduction. If possible let $f(x) \neq 0$ in (x_1, x_2) then there is at least one point x_0 in (x_1, x_2) such that $f(x_0) \neq 0$.

Then because of continuity of $f(x)$ in (x_1, x_2), there must exist an interval $(x_0 - \delta, x_0 + \delta)$ where $(\delta > 0)$ surrounding x_0 such that $f(x) > 0$ for all $x \in [x_0 - \epsilon, x_0 + \epsilon]$.

Since $g(x)$ is arbitrary it can be taken as

$$g(x) = \begin{cases} (x - x_0 + \delta)^2 (x - x_0 + \delta)^2 \\ \qquad x_0 - \delta \leq x \leq x_0 + \delta \\ 0 \quad ; \quad \text{otherwise} \end{cases}$$

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It is obvious that g vanishes at the end points of $(x_0-\delta, x_0+\delta)$ and has continous derivatives (because polynomial is always continous.)

By supposition $\int_{x_0-\delta}^{x_0+\delta} f(x)\cdot g(x)\cdot dx = 0$

But $\int_{x_0-\delta}^{x_0+\delta} f(x)(x-x_0-\delta)^2(x-x_0+\delta)^2\,dx > 0$

a contradiction.

Hence $f(x)=0 \quad \forall\, x \in (x_1, x_2)$

EULER-LAGRANGE EQUATION

Here we will derive a differential equation from which the required extremal curves or surface can be obtained.

In each case, we will extremize the functional of the form

$$T[y(x)] = \int_c F(x, y, y', y'' \text{---}) dx$$

THEOREM:- (For one dependent variable)

Statment:- let $I = \int_{x_1}^{x_2} F(x, y, y')\,dx$

where $y = y(x)$ is a continous first and second order derivatives satisfing the end conditions

$y(x_1) = y_1$ and $y(x_2) = y_2$

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If F is supposed to have continous first and second order derivatives then the function $y(x)$ will extremize the given integral if it satisfies the D-E

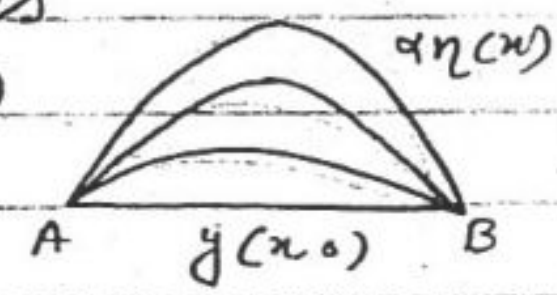
$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0}$$

Proof:- It is clear that between the fixed points $A(x_1, x_2)$ and $B(x_1, x_2)$ there are infinite number of curves infinite numbes of values to the functional $I[y(x)]$.

We have to finide that curve $y(x)$ which gives stationary value at I.

Let the family of all curves passing through A and B be defined as

$$y(x, \alpha) = y(x, 0) + \alpha \eta(x)$$



where $\eta(x)$ represents the deviation from the curve $y(x, 0)$ and α is the parameter lebelling different paths and is independent of x.

Suppose that the extremal corresponds to $\alpha = 0$, since all the curves passing through end points, So $\eta(x_1) = 0 = \eta(x_2)$

$$I[y] = \int_{x_1}^{x_2} F(x, y, y')\, dx$$

$$\frac{\partial I}{\partial \alpha} = \int_{\alpha_1}^{x_2} \frac{\partial}{\partial \alpha} F(x, y, y').\, dx$$

Since α and x all independent Therefoo

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$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial \alpha} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial \alpha} + \frac{\partial F}{\partial y'} \cdot \frac{\partial y'}{\partial \alpha} \right) dx \quad (1)$$

As α and x are independent $\frac{\partial x}{\partial \alpha} = 0$

As $y(x, \alpha) = y(x, 0) + \alpha \eta(x)$

$$\frac{\partial y}{\partial \alpha} = 0 + \eta(x)$$

And as $y'(x, \alpha) = y'(x, 0) + \alpha \eta'(x)$

$$\frac{\partial y'}{\partial \alpha} = \eta'(x)$$

putting in (1)

$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial x} \cdot 0 + \frac{\partial F}{\partial y} \cdot \eta(x) + \frac{\partial F}{\partial y'} \eta'(x) \right) dx$$

$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} \eta(x) + \frac{\partial F}{\partial y'} \eta'(x) \right) dx \quad (2)$$

As $\int_{x_1}^{x_2} \frac{\partial F}{\partial y'} \eta'(x)\, dx = \frac{\partial F}{\partial y'} \eta(x) \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x) \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx$

$$= \frac{\partial F}{\partial y'} \left[\eta(x_2) - \eta(x_1) \right] - \int_{x_1}^{x_2} \eta(x) \cdot \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx$$

$$= \frac{\partial F}{\partial y'} \left[0 - 0 \right] - \int_{x_1}^{x_1} \eta(x) \cdot \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx$$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'} \eta'(x) = 0 - \int_{x_1}^{x_2} \eta(x) \cdot \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx$$

putting in (2)

$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \right] \eta(x)\, dx = 0$$

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$$\text{or} \quad \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \right] \eta(x)\, dx = 0$$

By Fundamental theorem of calculus of variation.

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0}$$

SPECIAL CASES OF EULER LANGRANGE EQUATION

i) If F is independent of y

$$F = F(x, y') \quad , \quad F \neq F(y)$$

$$\frac{\partial F}{\partial y} = 0$$

Then Euler Lagranges equation becomes

$$-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

integrating on both sides

$$\frac{\partial F}{\partial y'} = a \text{ (Constant)}.$$

ii) If F does not depends explicity upon x.

$$F = F(y, y')$$

$$dF = \frac{\partial F}{\partial y} \cdot dy + \frac{\partial F}{\partial y'} \cdot dy' \quad \text{---} (1)$$

From Euler Langrange's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$$

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$$\frac{\partial F}{\partial y} = \frac{d}{dy}\left(\frac{\partial F}{\partial y'}\right)\frac{dy}{dx} \qquad \therefore \frac{d}{dx} = \frac{d}{dy}\cdot\frac{dy}{dx}$$

$$= \frac{d}{dy}\left(\frac{\partial F}{\partial y'}\right)y'$$

$$\frac{\partial F}{\partial y}dy = d\left(\frac{\partial F}{\partial y'}\right)y' \text{ ——— (2)}$$

using (2) in (1)

$$dF = d\left(\frac{\partial F}{\partial y'}\right)y' + \frac{\partial F}{\partial y'}dy'$$

$$dF = d\left(\frac{\partial F}{\partial y'}\cdot y'\right)$$

$$dF - d\left(\frac{\partial F}{\partial y'}\cdot y'\right) = 0$$

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$$\Rightarrow d\left[F - \frac{\partial F}{\partial y'}\cdot y'\right] = 0$$

integrating on both sides

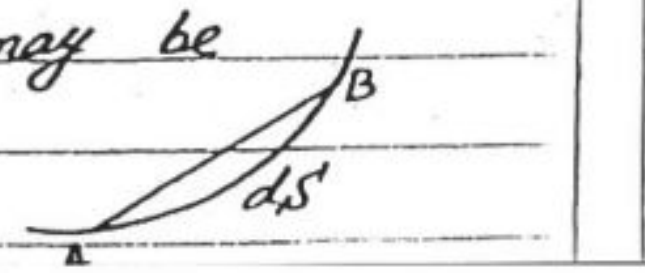
$$\boxed{F - \frac{\partial F}{\partial y'}\cdot y' = \alpha \text{ (Constant)}}$$

A/2015, A/2013

GEODESIC PROBLM

Statment:- To find the curve whose distance between two points on a surface is minimum.

Proof:- Let A and B be any points on the curve C, laying on a surface S. The equation of the surface may be taken as $Z = Z(x, y)$.



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The distance b/w the points A and B on any curve $y = y(x)$ is given by

$$l = \int_A^B ds$$

$$= \int_A^B \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

$$= \int_A^B \sqrt{(dx)^2\left[1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2\right]}$$

$$= \int_A^B \sqrt{1 + (y')^2 + (z')^2}\ .\ dx$$

Here ds is the element of arc along the curve.

In XY-plane, the expression for l becomes

$$l = \int_A^B \sqrt{1 + (y')^2}\ .\ dx$$

$$F(x, y, y') = \sqrt{1 + (y')^2}$$

F is independent of y

There for $\frac{\partial F}{\partial y'} = \alpha$ (constant)

$$\frac{\partial [1 + y'^2]^{1/2}}{\partial y'} = \alpha$$

$$\frac{1}{2}[1 + y'^2]^{-1/2} \cdot (2y') = \alpha$$

$$\frac{y'}{\sqrt{1 + y'^2}} = \alpha$$

$$y' = \alpha\left[\sqrt{1 + y'^2}\right]$$

Squaring both side

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$$y'^2 = \alpha^2[1+y'^2] = \alpha^2 + \alpha^2 y'^2$$

$$y'^2 - \alpha^2 y'^2 = \alpha^2$$

$$y'^2[1-\alpha^2] = \alpha^2$$

$$y'^2 = \frac{\alpha^2}{1-\alpha^2}$$

$$y' = \pm \frac{\alpha}{\sqrt{1-\alpha^2}}$$

$$y' = \pm c \quad \text{where } c = \frac{\alpha}{\sqrt{1-\alpha^2}}$$

Integrating both side

$$y = \pm cx + d$$

Hence $y(x)$ is a straight line.

*** ***

A/2013, A/2010, A/2008

Written by
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BRACHISTOCHRONE PROBLEM

Q :- Stat and Poove Brachistochoone Pooblem.

Statment :- A Partical fall froom (under gravity) one point to another. Determine the curve alony which time taken by partical will be minimum.

Anothes Statment :- To find the equetion of the path in space down which a partical will fall under the gravity from one point to another in the shortest possible time.

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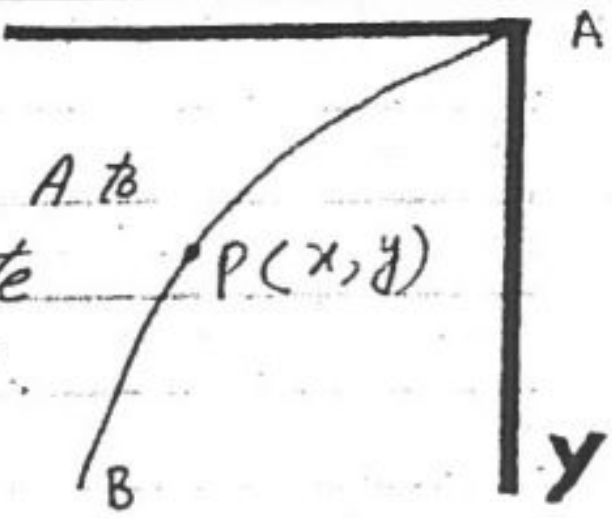
Proof :- Let a particle fall from a point A to another point B. There are infinite number of paths b/w A and B. But we choose that path only along which time taken is minimum.



Total time taken by the partical in falling from A to B is given by

$$T = \text{total time} = \int_A^B dt$$

$$= \int_A^B \frac{dt}{ds} \times ds$$

Written by
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$$= \int_A^B \frac{1}{v} \cdot ds \quad \text{— (1)} \quad \left(\because v = \frac{ds}{dt}\right)$$

As $\frac{1}{2} mv^2 = mgh$

$$v^2 = 2gh \Rightarrow v = \sqrt{2gh} \quad \text{— (2)}$$

Also $(ds)^2 = (dx)^2 + (dy)^2$

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\frac{ds}{dx} = \sqrt{1+y'^2} \Rightarrow ds = \sqrt{1+y'^2}\, dx \quad \text{— (3)}$$

using (2) and (3) in (1), we get

$$T = \int_A^B \frac{1}{\sqrt{2gy}} \cdot \sqrt{1+y'^2}\, dx$$

$$= \frac{1}{\sqrt{2g}} \int_A^B \sqrt{\frac{1+y'^2}{y}}\, dx$$

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Here $F = \sqrt{\frac{1+y'^2}{y}}$

Since F is indepnt of x, E-L equation reduce to

$$F - y' \frac{\partial F}{\partial y'} = c \quad (\text{Constant})$$

$$\sqrt{\frac{1+(y')^2}{y}} - y' \frac{\partial}{\partial y'} \sqrt{\frac{1+(y')^2}{y}} = c$$

$$\sqrt{\frac{1+(y')^2}{y}} - y' \frac{1}{\sqrt{y}} \cdot \frac{1}{2\sqrt{1+y'^2}} \cdot 2y' = c$$

$$\sqrt{\frac{1+(y')^2}{y}} - \frac{(y')^2}{\sqrt{y(1+y'^2)}} = c$$

$$\frac{1}{\sqrt{y}} \left[\sqrt{1+(y')^2} - \frac{(y')^2}{\sqrt{1+y'^2}} \right] = c$$

$$\frac{1}{\sqrt{y}} \left[\frac{[1+(y')^2] - (y')^2}{\sqrt{1+y'^2}} \right] = c$$

$$\frac{1}{\sqrt{y}} \left[\frac{1}{\sqrt{1+y'^2}} \right] = c$$

Squaring both side

$$\frac{1}{y(1+(y')^2)} = c^2$$

$$\frac{1}{y} = c^2 [1+(y')^2]$$

$$\frac{1}{c^2 y} = [1+(y')^2]$$

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$$(y')^2 = \frac{1}{c^2y} - 1 = \frac{1 - c^2y}{c^2y}$$

$$\frac{dy}{dx} = \sqrt{\frac{1-c^2y}{c^2y}}$$

$$\frac{dx}{dy} = \sqrt{\frac{c^2y}{1-c^2y}}$$

Written by
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$$dx = \sqrt{\frac{c^2y}{1-c^2y}}\, dy \quad \text{---(4)}$$

Let $c^2y = \sin^2\frac{\theta}{2}$

$$y = \frac{1}{c^2}\sin^2\theta/2$$

$$dy = \frac{1}{c^2}\left[2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}\right]\cdot\frac{1}{2}\,d\theta$$

$$dy = \frac{1}{c^2}\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}\cdot d\theta$$

Putting in (4)

$$dx = \sqrt{\frac{\sin^2\theta/2}{1-\sin^2\theta/2}}\cdot\frac{1}{c^2}\cdot\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}\cdot d\theta$$

$$dx = \sqrt{\frac{\sin^2\theta/2}{\cos^2\theta/2}}\cdot\frac{1}{c^2}\cdot\sin\theta/2\cdot\cos\theta/2\cdot d\theta$$

$$dx = \frac{\sin\theta/2}{\cos\theta/2}\cdot\frac{1}{c^2}\cdot\sin\theta/2\cdot\cos\theta/2\cdot d\theta$$

$$dx = \frac{1}{c^2}\sin^2\theta/2\cdot d\theta$$

$$dx = \frac{1}{c^2}\left[\frac{1-\cos\theta}{2}\right]d\theta$$

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$$dx = \frac{1}{2C^2}(1-\cos\theta)\,d\theta$$

integrating

$$x = \frac{1}{2C^2}[\theta - \sin\theta] + \text{Constant}$$

And

$$y = \frac{1}{2C^2}(1-\cos\theta)$$

A/2008

Question 1:- Find the curve joining the points (x_1, y_1) and (x_2, y_2), which gives minimum area of surface of revolution around x-axis

Solution:-

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points in xy-plane.

We find the curve which give gives minimum area of suface about x-axis

Let the curve revolues about x-axis

$$\text{Area} = \int_A^B 2\pi y\, ds$$

$$= 2\pi \int_A^B y \cdot \sqrt{(dx)^2 + (dy)^2}$$

$$= 2\pi \int_A^B y \sqrt{1 + y'^2}\, dx$$

Here $F = y\sqrt{1+y'^2}$

Since F is independent of x, So E-L equation reduces to

$$F - y' \frac{\partial F}{\partial y'} = \text{constant}$$

$$y\sqrt{1+(y')^2} - y' \frac{\partial}{\partial y'}\left[y\sqrt{1+(y')^2}\right] = \text{constant}$$

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$$y\sqrt{1+(y')^2} - y'\cdot y\cdot \frac{1}{2}\left[1+(y')^2\right]^{-1/2}\cdot 2y' = C$$

$$y\sqrt{1+(y')^2} - \frac{y(y')^2}{\sqrt{1+(y')^2}} = C$$

$$\frac{y(1+(y')^2) - y(y')^2}{\sqrt{1+(y')^2}} = C$$

$$\frac{y + \cancel{y(y')^2} - \cancel{y(y')^2}}{\sqrt{1+(y')^2}} = C$$

$$\frac{y}{\sqrt{1+(y')^2}} = C$$

$$\frac{y}{c} = \sqrt{1+y'^2}$$

$$\frac{y^2}{c^2} = 1+y'^2$$

$$y'^2 = \frac{y^2}{c^2} - 1 = \frac{y^2-c^2}{c^2}$$

$$\Rightarrow y' = \frac{\sqrt{y^2-c^2}}{c}$$

$$\frac{dy}{dx} = \frac{\sqrt{y^2-c^2}}{c}$$

Written by
Labeeb Ahmad

$$\frac{dx}{dy} = \frac{c}{\sqrt{y^2-c^2}}$$

$$dx = \frac{c}{\sqrt{y^2-c^2}}\,dy$$

integrating $\int dx = \int \frac{c}{y^2-c^2}\,dy$

$$x = c\,\cosh^{-1}(y/c) + \text{Constant}$$

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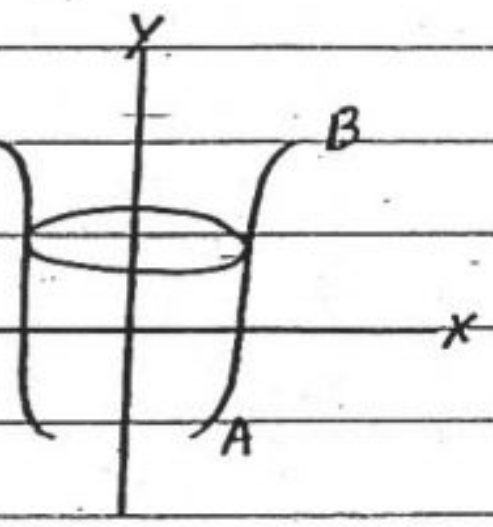
A/2007

Question 2 :- Find the curve joining the points (x_1, y_1) and (x_2, y_2) which give minimum area of surface of revolution around y-axis.

Solution :- Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points in x-y plane.

we find the curve which gives minimum area of surface of revolution.

Let the curve revolves about y-axis



$$A = \int_A^B 2\pi x \, dS$$

$$= \int_A^B 2\pi x \sqrt{(dx)^2 + (dy)^2}$$

$$= 2\pi \int_A^B x \sqrt{(dx)^2 \left[1 + \left(\frac{dy}{dx}\right)^2\right]}$$

$$= 2\pi \int_A^B x \sqrt{1 + (y')^2} \cdot dx$$

Here $F = x\sqrt{1+(y')^2}$.

Since F is independent of y. So Euler Langrang equation reduces

$$\frac{\partial F}{\partial y'} = a$$

$$\frac{\partial}{\partial y'}\left[x\sqrt{1+(y')^2}\right] = a$$

$$x \frac{1}{2\sqrt{1+(y')^2}} 2y' = a$$

$$xy' = a\sqrt{1+(y')^2}$$

$$(xy')^2 = a^2\left[1+(y')^2\right]$$

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$$x^2(y')^2 = a^2 + a^2(y')^2$$

$$x^2(y')^2 - a^2(y')^2 = a^2$$

$$(x^2 - a^2)y'^2 = a^2$$

$$y' = \frac{a}{\sqrt{x^2 - a^2}}$$

$$\frac{dy}{dx} = \frac{a}{\sqrt{x^2 - a^2}}$$

Written by
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$$dy = \frac{a}{\sqrt{x^2 - a^2}} \cdot dx$$

Integrating both side

$$\int dy = \int \frac{a}{\sqrt{x^2 - a^2}} \cdot dx$$

$$y = a\cosh^{-1}\frac{x}{c} + \text{Constant}$$

1/2010, 5/2009

DIDO'S PROBLEM

Question :- Find a curve of given length which enclose maximum area with x-axis :

Q :- Stat and poove Dido's Pooblem.

tatment :- To find a closed curve $y = y(x)$ of given length l which encloses maximum area.

Proof :- Suppose that $y = y(x)$ is the curve which meets the x-axis at points $A(x_1, 0)$ and $B(x_2, 0)$ and encloses max area.

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Area enclosed $= \int_{x_1}^{x_2} y\,dx$

Therefore we determine ∝ we have to extremize the functional

$$I[y] = \int_{x_1}^{x_2} y\,dx$$

Subject to the condition

$$l = \int_{x_1}^{x_2} ds$$

$$l = \int_{x_1}^{x_2} \sqrt{(dx)^2 + (dy)^2}$$

$$l = \int_{x_1}^{x_2} \sqrt{(dx)^2\left[1 + \left(\frac{dy}{dx}\right)^2\right]}$$

$$l = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\,dx$$

Here $F = y$ and $G = \sqrt{1+y'^2}$

Therefor

$$H = F + \lambda G$$

$$H = y + \lambda\sqrt{1+y'^2}$$

Since there is no explicit dependence on x we use Euler Lagrange equation

$$H - y'\frac{\partial H}{\partial y'} = \text{Constant}$$

$$y + \lambda\sqrt{1+(y')^2} - y'\frac{\partial}{\partial y'}\left[y + \lambda\sqrt{1+y'^2}\right] = \text{Cons}$$

$$y + \lambda\sqrt{1+(y')^2} - y'\left[0 + \lambda \frac{2y'}{2\sqrt{1+y'^2}}\right] = \text{Constan}$$

$$y + \lambda\sqrt{1+(y')^2} - \frac{\lambda(y')^2}{\sqrt{1+(y')^2}} = \text{constant}$$

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$$\frac{y\sqrt{1+(y')^2} + \lambda(1+y'^2) - \lambda y'^2}{\sqrt{1+(y')^2}} = c_1$$

$$\frac{y\sqrt{1+(y')^2} + \lambda + \cancel{\lambda(y')^2} - \cancel{\lambda(y')^2}}{\sqrt{1+(y')^2}} = c_1$$

$$y + \frac{\lambda}{\sqrt{1+(y')^2}} = c_1$$

Written by
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$$\frac{\lambda}{\sqrt{1+(y')^2}} = c_1 - y$$

$$\frac{\lambda}{c_1 - y} = \sqrt{1+(y')^2}$$

Squaring on both sides

$$\frac{\lambda^2}{(c_1-y)^2} = 1 + (y')^2$$

$$(y')^2 = \frac{\lambda^2}{(c_1-y)^2} - 1$$

$$(y')^2 = \frac{\lambda^2 - (c_1-y)^2}{(c_1-y)^2}$$

$$y' = \sqrt{\frac{\lambda^2 - (c_1-y)^2}{(c_1-y)^2}}$$

$$\frac{dy}{dx} = \sqrt{\frac{\lambda^2 - (c_1-y)^2}{(c_1-y)^2}}$$

$$\frac{dx}{dy} = \frac{(c_1-y)}{\sqrt{\lambda^2-(c_1-y)^2}}$$

$$dx = \frac{(c_1-y)}{\sqrt{\lambda^2-(c_1-y)^2}}\, dy$$

Integrating on both sides

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$$\int dx = \int \frac{(c_1 - y)}{\sqrt{\lambda^2 - (c_1 - y)^2}}\, dy$$

Let $c_1 - y = z$

$dy = -dz$

$$x + c_2 = \int \frac{z}{\sqrt{\lambda^2 - z^2}}(-dz)$$

$$= \frac{1}{2}\int (\lambda^2 - z^2)^{-\frac{1}{2}} (-2z\, dz)$$

$$= \frac{1}{2} \frac{(\lambda^2 - z^2)^{\frac{1}{2}}}{\frac{1}{2}}$$

$$x + c_2 = \sqrt{\lambda^2 - z^2}$$

$$(x + c_2)^2 = \lambda^2 - z^2$$

$$(x + c_2)^2 + z^2 = \lambda^2$$

$$(x + c_2)^2 + (c_1 - y)^2 = \lambda^2$$

$$(x + c_2)^2 + (y - c_1)^2 = \lambda^2$$

which is the equation of circular arc.

Constrained Extrema :-

The problem is to find the extremal curves $y = y(x)$ for $I[y] = \int_{x_1}^{x_2} F(x, y, y')\, dx$ with end point conditions

$y(x_1) = y_1$ and $y(x_2) = y_2$

And

$$I[y] = \int_{x_1}^{x_2} G(x, y, y')\, dx$$

Then $H = F + \lambda G$

So E-L equation is $\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$

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A/2016, A/2011, A/2010, S/2010

THEOREM:- Show that a solid of revolution for a given surface area having maximum volume, is a sphere.

OR

Find the curve which generates a surface of revolution of given area which enclose maximum volume.

Solution:-

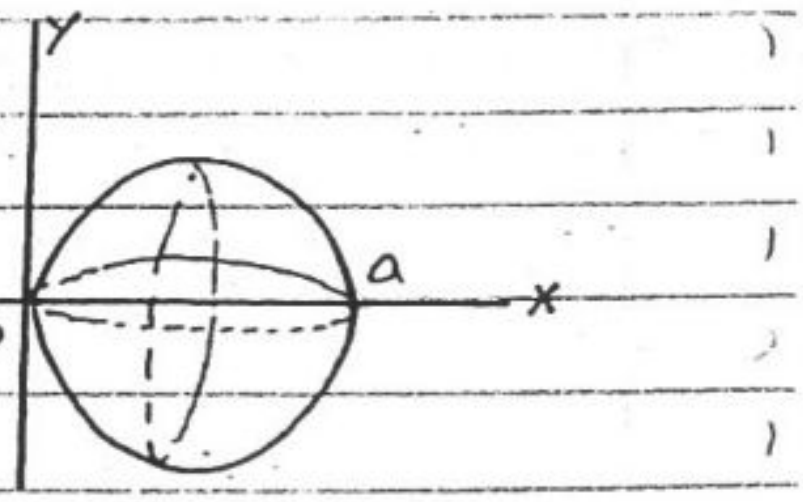
Let the curve $y = y(x)$ with $y(0) = 0$, $y(a) = 0$ be rotated about x-axis so as generates a surface of revolution.

An element of surface area A is $\pi y (dS)$. Therefor, Total surface area A is given by

$$A = \int_0^a 2\pi y \, dS$$

$$= \int_0^a 2\pi y \sqrt{(dx)^2 + (dy)^2}$$

$$= 2\pi \int_0^a y \sqrt{1 + (y')^2} \, dx$$



The volume of element of solid revolution $= \pi y^2 dx$

So

$$\text{Total volume} = \int_0^a \pi y^2 \, dx$$

Thus we have to find the curve $y = y(x)$ with

$$y(0) = 0, \quad y(a) = 0$$

which maximize $I[y] = \int_0^a \pi y^2 \, dx$

subject to

$$A[y] = 2\pi \int_0^a y \sqrt{1 + (y')^2} \, dx$$

In this case

$$F = y^2, \quad G = y\sqrt{1 + (y')^2}$$

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Hence $H = F + \lambda G$

$= y^2 + \lambda y\sqrt{1+(y')^2}$

where overall constant π is unimportant and other constants has been absorbed in λ. Since H does not depend on x. So E-L equation can be taken as

$$H - y' \frac{\partial H}{\partial y'} = \text{Constant}$$

$$y^2 + \lambda y\sqrt{1+(y')^2} - y' \frac{\partial}{\partial y'}\left[y^2 + \lambda y\sqrt{1+y'^2}\right] = c$$

$$y^2 + \lambda y\sqrt{1+(y')^2} - y'\left[\lambda y \frac{(2y')}{2\sqrt{1+y'^2}}\right] = c$$

$$y^2 + \lambda y\sqrt{1+(y')^2} - \frac{\lambda y y'^2}{\sqrt{1+y'^2}} = c$$

$$y^2\left[1+y'^2 + \lambda y(1+y'^2) - \lambda y y'^2\right] = c$$

$$\sqrt{1+y'^2}$$

$$y^2\sqrt{1+y'^2} + \lambda y + \lambda y y'^2 - \lambda y y'^2 = c$$

$$\sqrt{1+y'^2}$$

$$y^2\sqrt{1+y'^2} + \lambda y = c\sqrt{1+y'^2}$$

$$\lambda y = c\sqrt{1+y'^2} - y^2\sqrt{1+y'^2}$$

$$\lambda y = (c - y^2)\sqrt{1+y'^2}$$

Now $y(0) = 0 \Rightarrow c = 0$

$$\lambda y = (0 - y^2)\sqrt{1+y'^2}$$

Squaring both side

$$\lambda^2 y^2 = y^4(1+y'^2)$$

$$\lambda^2 = y^2 + y^2 y'^2$$

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$$y^2(y')^2 = \lambda^2 - y^2$$

$$(y')^2 = \frac{\lambda^2 - y^2}{y^2}$$

$$y' = \frac{\sqrt{\lambda^2 - y^2}}{y}$$

$$\frac{dy}{dx} = \frac{\sqrt{\lambda^2 - y^2}}{y}$$

Written by
Labeeb Ahmad

$$\frac{dx}{dy} = \frac{y}{\sqrt{\lambda^2 - y^2}}$$

$$dx = \frac{y}{\sqrt{\lambda^2 - y^2}} \cdot dy$$

integrating on both side

$$\int dx = \int \frac{y}{\sqrt{\lambda^2 - y^2}} \cdot dy$$

$$= -\frac{1}{2}\int (\lambda^2 - y^2)^{-1/2} (-2y \cdot dy)$$

$$x = -\frac{1}{2}\frac{(\lambda^2 - y^2)^{1/2}}{1/2} + 4$$

$$x = -\sqrt{\lambda^2 - y^2} + 4$$

$$x - 4 = -\sqrt{\lambda^2 - y^2}$$

$$(x-4)^2 = \lambda^2 - y^2 \Rightarrow (x-4)^2 + y^2 = \lambda^2$$

which is the equation of circle at $(4, 0)$ and radius λ.

This show that the curve that generates a surface revolution of given surface but of maximum volume is a circle.

Hence surface of revolution will be a sphere.

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Question:- Prove that $\delta I = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} \cdot \delta y + \frac{\partial F}{\partial y'} \cdot \delta y' \right)$

Solution:- $\frac{\delta I}{\delta \alpha} \approx \frac{\partial I}{\partial \alpha}$ $\left(\because \frac{\delta y}{\delta x} \approx \frac{dy}{dx} \right)$

$$\delta I \approx \frac{\partial I}{\partial \alpha} \times \delta \alpha \quad \text{---(1)}$$

$$I[y] = \int_{x_1}^{x_2} F(x, y, y') \cdot dx$$

$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial \alpha} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial \alpha} + \frac{\partial F}{\partial y'} \cdot \frac{\partial y'}{\partial \alpha} \right) \cdot dx$$

putting in (1)

$$\delta I = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial \alpha} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial \alpha} + \frac{\partial F}{\partial y'} \cdot \frac{\partial y'}{\partial \alpha} \right) dx \, \delta \alpha \quad \text{(2)}$$

As α is independent of x so $\frac{\partial x}{\partial \alpha} = 0$

Also as $y(x, \alpha) = y(x, 0) + \alpha \eta(x)$

$$\frac{\partial y}{\partial \alpha} = \eta(x)$$

As $y'(x, \alpha) = y'(x, 0) + \alpha \eta'(x)$.

$$\frac{\partial y'}{\partial \alpha} = \eta'(x)$$

putting in (2)

$$\delta I \approx \int_{x_1}^{x_2} \left[0 + \frac{\partial F}{\partial y} \cdot \eta(x) + \frac{\partial F}{\partial y'} \eta'(x) \right] \delta \alpha \cdot dx$$

$$\delta I \approx \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} \eta(x) \, \delta \alpha + \frac{\partial F}{\partial y'} \eta'(x) \cdot \delta \alpha \right] dx \quad \text{---(3)}$$

Now $\frac{\delta y}{\delta \alpha} \approx \frac{\partial y}{\partial \alpha} \Rightarrow \delta y \approx \frac{\partial y}{\partial \alpha} \cdot \delta \alpha$

$$\delta y \approx \eta(x) \, \delta \alpha \quad \left(\because \frac{\partial y}{\partial \alpha} = \eta(x) \right)$$

Similarly $\delta y' \approx \eta'(x) \, \delta \alpha$

putting in (3)

$$\delta I \approx \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} \cdot \delta y + \frac{\partial F}{\partial y'} \delta y' \right) dx$$

Hence prove

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12014, A/2011, A/2009, S/2009

EULER LANGRANGE equation for one independent and n-Dependent Variable

Question:- Stat and Prove Euler - Langrange equation for n-dependent variable.

Statment:- Let $I = \int_{x_1}^{x_2} F(x, y_1, y_2', y_3 \text{---} y_n, y_1', y_2' \text{---} y_n')\,dx$

where $y_K = y_K(x)$ is a continous function having continous first and second order derivatives satisfing the following boundary conditions

$y_K(x_1) = \text{Constant}$, $y_K(x_2) = \text{Constant}$ $K = 1, 2 \text{---} n$

If F is supposed to be continous first and second order derivatives then the function $y_K(x)$ will extremize the given integral if it satisfies the the D-E $\frac{\partial F}{\partial y_K} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_K'}\right) = 0$, $K = 1, 2 \text{---} n$

Proof:- Using the short hand procedure for finding the variation δI of I , when y_K varies by δy_K , $K = 1, 2, 3 \text{---} n$

we have $\delta I = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y_1} \cdot \delta y_1 + \frac{\partial F}{\partial y_2} \delta y_2 + \text{---} \frac{\partial F}{\partial y_n} \cdot \delta y_n \right] +$

$\left[\frac{\partial F}{\partial y_1'} \delta y_1' + \frac{\partial F}{\partial y_2'} \delta y_2' + \text{---} \frac{\partial F}{\partial y_n'} \delta y_n' \right] dx$

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or

$$\delta I = \int_{x_1}^{x_2} \left[\left(\frac{\partial F}{\partial y_1} \delta y_1 + \frac{\partial F}{\partial y_1'} \delta y_1' \right) + \left(\frac{\partial F}{\partial y_2} \delta y_2 + \frac{\partial F}{\partial y_2'} \delta y_2' \right) + \text{——} + \left(\frac{\partial F}{\partial y_n} \delta y_n + \frac{\partial F}{\partial y_n'} \delta y_n' \right) \right] dx \quad \text{(1)}$$

Now Consider

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y_k} \delta y_k + \frac{\partial F}{\partial y_k'} \delta y_k' \right] dx$$

$$= \int_{x_1}^{x_2} \frac{\partial F}{\partial y_k} \delta y_k \, dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial y_k'} \delta y_k' \, dx$$

$$= \int_{x_1}^{x_2} \frac{\partial F}{\partial y_k} \delta y_k \, dx + \left[\frac{\partial F}{\partial y_k'} \delta y_k \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial F}{\partial y_k'} \right) \delta y_k \, dx \right]$$

$$= \int_{x_1}^{x_2} \frac{\partial F}{\partial y_k} \delta y_k \, dx + 0 - \int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial F}{\partial y_k'} \right) \delta y_k \, dx$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y_k} - \frac{d}{dx} \left(\frac{\partial F}{\partial y_k'} \right) \right] \delta y_k \, dx \qquad \left(\because \delta y_k = 0 \text{ at } x = x_1, x_2 \right)$$

These for equation (1) becomes

$$\delta I = \int_{x_1}^{x_2} \left(\left[\frac{\partial F}{\partial y_1} - \frac{d}{dx} \left(\frac{\partial F}{\partial y_1'} \right) \right] \delta y_1 + \left[\frac{\partial F}{\partial y_2} - \frac{d}{dx} \left(\frac{\partial F}{\partial y_2'} \right) \right] \delta y_2 + \text{——} + \left[\frac{\partial F}{\partial y_n} - \frac{d}{dx} \left(\frac{\partial F}{\partial y_n'} \right) \right] \delta y_n \right) dx$$

$$\delta I = \int_{x_1}^{x_2} \sum_{k=1}^{n} \left[\frac{\partial F}{\partial y_k} - \frac{d}{dx} \left(\frac{\partial F}{\partial y_k'} \right) \right] \delta y_k \cdot dx$$

The extremal corresponds to $\delta I = 0$

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Therefor $\int_{x_1}^{x_2} \sum_{K=1}^{n} \left[\frac{\partial F}{\partial y_K} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_K'} \right) \right] \delta y_K \, dx = 0$ — (2)

Since the curves y_K are independent of each other, By fundamental theorem of calculus of variation each of the integral on R.H.S of equation (2) must be zero

So $$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y_K} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_K'} \right) \right] \delta y_K \, dx = 0$$

Now involving the fundamental theorem of calculus of variation, we have

$$\boxed{\frac{\partial F}{\partial y_K} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_K'} \right) = 0}$$

A/2013

Question :- Derive E-L Equation of $I[y] = \int_{x_1}^{x_2} F(x, y, y', y'') \, dx$

Solution :- $I[y] = \int_{x_1}^{x_2} F(x, y, y', y'') \, dx$

with end point conditions

$y^{(K)}(x_1) = \text{constant}$, $y^{K}(x_2) = \text{Constant}$

$K = 1, 2$

using variation of I w.r.t α :

$$\delta I \approx \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' + \frac{\partial F}{\partial y''} \delta y'' \right] dx \quad \text{— (1)}$$

Consider $\int_{x_1}^{x_2} \frac{\partial F}{\partial y'} \delta y' \, dx = \frac{\partial F}{\partial y'} \delta y \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y} \right) \delta y \, dx$

$\therefore y(x_1) = \text{Constant}$, $y(x_2) = \text{Constant}$

So $\delta I = 0$

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$$= 0 - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \delta y \cdot dx \quad \text{---(ii)}$$

Similarly

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y''} \delta y'' \cdot dx = \frac{\partial F}{\partial y''} \delta y' \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right) \delta y' dx$$

$$= 0 - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right) \delta y' dx$$

$$= -\left[\frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right) \delta y \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \delta y \cdot \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) dx\right]$$

$$= -\left[0 - \int_{x_1}^{x_2} \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) \delta y \cdot dx\right]$$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y''} \delta y'' \cdot dx = (-1)^2 \int_{x_1}^{x_2} \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) \delta y \cdot dx \quad \text{---(iii)}$$

putting the value of (ii) and (iii) in (i)

$$\delta I \approx \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} \cdot \delta y + (-1)\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \delta y + (-1)^2 \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) \delta y\right] dx$$

For extremal value $\Rightarrow \quad \delta I = 0$

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} + (-1)\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + (-1)^2 \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)\right] \delta y \, dx = 0$$

By the Fundamental Theorem of calculus of variation, we get

$$\frac{\partial F}{\partial y} + (-1)\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + (-1)^2 \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0$$

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From
Sharaq pur Sharif

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Euler Langrange Equation for Higher Order Derivatives.

Solution:- $I[y] = \int_{x_1}^{x_2} F(x, y, y', y'' \cdots y^{(n)})\,dx$

with end points conditions

$y^{(K)}(x_1) = \text{Constant}$, $y^{(K)}(x_2) = \text{Constant}$

$K = 1, 2, 3 \cdots n$

using variation of I w.r.t α

$$\delta I \approx \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' + \frac{\partial F}{\partial y''}\delta y'' + \cdots + \frac{\partial F}{\partial y^{(n)}}\delta y^{(n)} \right] dx \quad -(1)$$

Consider $\int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\delta y'\,dx = \frac{\partial F}{\partial y'}\cdot\delta y\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y\,dx$

$\because y(x_1) = \text{Constant}$, $y(x_2) = \text{Constant}$

Therefor $\delta I = 0$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\delta y'\,dx = 0 - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y\cdot dx$$

$$= (-1)\int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y\cdot dx \quad -(2)$$

Similarly

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y''}\delta y''\cdot dx = \frac{\partial F}{\partial y''}\delta y'\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right)\delta y'\,dx$$

$$= 0 - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right)\delta y'\,dx$$

$$= -\left[\frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right)\delta y\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \delta y\,\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)dx\right]$$

$$= -\left[0 - \int_{x_1}^{x_2} \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)\delta y\cdot dx\right.$$

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A/2015

Question 2 :- Find the extremal of the functional.

$$I[y] = \int_0^1 (y^2 + y'^2)\, dx \quad , \quad y(0) = 0, \quad y(1) = 1$$

Solution :- $F = y^2 + y'^2$

By E-L Equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

$$2y - \frac{d}{dx}(2y') = 0$$

$$\therefore \quad 2y - 2y'' = 0$$

$$-2(y'' - y) = 0$$

$$y'' - y = 0 \quad , \quad (-2 \neq 0)$$

$$(D^2 - 1)y = 0$$

For non-trivial solution $y \neq 0$

So $D^2 - 1 = 0$

$$D = \pm 1$$

$$y = Ae^x + Be^{-x}$$

At $y(0) = 0$

$$0 = Ae^0 + Be^0$$

$$A + B = 0 \quad \text{---} (1)$$

At $y(1) = 1 \Rightarrow 1 = Ae^1 + Be^{-1}$ ---- (2)

From (1) $B = -A$ ---- (3)

Put in (2)

$$Ae - Ae^{-1} = 1$$

$$A(e - e^{-1}) = 1$$

$$A = \frac{1}{e - e^{-1}} = \frac{1}{\left(\frac{e - e^{-1}}{2}\right)2}$$

$$A = \frac{1}{2\sinh(1)}$$

Put in (3) $\quad B = \frac{-1}{2\sinh(1)}$

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Hence $y = \frac{1}{2\sinh(1)} e^{x} - \frac{1}{2\sinh(1)} e^{-x}$

$$y = \frac{1}{\sinh(1)} \left[\frac{e^{x} - e^{-x}}{2} \right]$$

$$y = \frac{\sinh x}{\sinh(1)}$$

A/2011, A/2010, S/2009, S/2010

Question 3:- Find the stationary values for integral

$$\int_{A(0,\pi)}^{B(\pi,0)} (2y\sin x + y'^{2}) \, dx$$

Solution:- Let $I[y] = \int_{A(0,\pi)}^{B(\pi,0)} (2y\sin x + y'^{2}) dx$ — (1)

Here $F = 2y\sin x + y'^{2}$

The function $y(x)$ will extremize given integral if it satisfies the E-L equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \text{— (2)}$$

$$\frac{\partial}{\partial y}[2y\sin x + y'^{2}] - \frac{d}{dx}\left[\frac{\partial}{\partial y'}(2y\sin x + y'^{2})\right] = 0$$

$$2\sin x - \frac{d}{dx}(0 + 2y') = 0$$

$$2\sin x - 2y'' = 0$$

$$2y'' = 2\sin x$$

$$y'' = \sin x$$

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$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y''}\delta y''.dx = (-1)^2 \int_{x_1}^{x_2} \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)\delta y\, dx \quad -(3)$$

Similarly $$\int_{x_1}^{x_2} \frac{\partial F}{\partial y^{(n)}}\delta y^{(n)}.dx = (-1)^n \int_{x_1}^{x_2} \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right)\delta y\, dx \quad -(4)$$

putting the value of (2), (3) and (4) in (1)

$$\delta I \approx \int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y}\delta y + (-1)\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y + (-1)^2\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)\delta y\right.$$

$$\left. + \text{——————} + (-1)^n\frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^n}\right)\delta y\right]dx$$

For extremal

$$\Rightarrow \delta I = 0$$

$$\int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y} + (-1)\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + (-1)^2\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) + \text{——}\right.$$

$$\left. + \text{——————} + (-1)^n\frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right)\right]\delta y\, dx = 0$$

By the Fundamental Theorem of calculus of variation, we get

$$\frac{\partial F}{\partial y} + (-1)\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + (-1)^2\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) + \text{——————}$$

$$\text{——————} + (-1)^n\frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) = 0$$

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Question 1 :- Find the extremal for functional

$$I[y] = \int_0^1 (x + y'^2)\, dx \quad , \quad y(0) = 1$$
$$y(1) = 2$$

Solution :- Here

$$F = x + y'^2$$

F is independent of y

$$\text{So} \quad \frac{\partial F}{\partial y'} = \text{constant}$$

$$\frac{\partial}{\partial y'}(x + y'^2) = c$$

$$0 + 2y' = c$$

$$y' = \frac{c}{2}$$

integrating

$$y = \frac{c}{2}x + d$$

Applying given conditions

$$y(0) = 1$$

$$\frac{c}{2}(0) + d = 1 \Rightarrow \boxed{d = 1}$$

At $y(1) = 2$

$$\frac{c}{2}(1) + d = 2$$

$$\frac{c}{2} + 1 = 2 \qquad (\because d = 1)$$

$$\frac{c}{2} = 2 - 1$$

$$\boxed{c = 2}$$

Hence $y = x + 1$

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integrating both side

$$y' = -\cos x + A$$

$$\Rightarrow y = -\sin x + Ax + B$$

where A and B are constants.

Apply given Conditions

At $A(0, \pi)$, we have

$$\pi = -\sin(0) + A(0) + B$$

$$\Rightarrow \boxed{B = \pi}$$

At $B(\pi, 0)$, we have

$$0 = -\sin\pi + A\pi + B$$

$$0 = 0 + A\pi + \pi$$

$$0 = (A+1)\pi$$

Either $A+1 = 0$ or $\pi = 0$

But $\pi \neq 0$, so $A + 1 = 0 \Rightarrow \boxed{A = -1}$

Hence

$$y = -\sin x - x + \pi$$

$$y = \pi - x - \sin x$$

S/2010

Question 4:- Find the extremal curve for functional.

$$I[y] = \int_{A(0,0)}^{A'(1,2)} [y'^2 + (1+y')^2]\,dx$$

Solution:-

$$F = y'^2 + (1+y')^2$$

$$F = y'^2 + 1 + y'^2 + 2y'$$

$$F = 2y'^2 + 2y' + 1$$

Since F is independent of y, the E-L equation is reduced.

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$$\frac{\partial F}{\partial y'} = \text{Constant}$$

$$4y' + 2 = C$$

$$y' = \frac{C-2}{4}$$

integrating

$$y = \left(\frac{C-2}{4}\right)x + d$$

At $A(0,0)$, we get

$$0 = 0 + d \Rightarrow \boxed{d=0}$$

At $A'(1,2)$, we get

$$2 = \left(\frac{C-2}{4}\right)(1) + d$$

$$2 = \left(\frac{C-2}{4}\right) + 0$$

$$8 = C-2 \Rightarrow \boxed{C=10}$$

Hence, the required extremal functional is

$$y = \left(\frac{10-2}{4}\right)x + 0$$

$$y = \left(\frac{8}{4}\right)x$$

$$y = 2x$$

A/2007

Question 5 :- From among the curves connecting the points $A(1,3)$ and $B(2,5)$. Find the extremal curve of the functional

$$I[y] = \int_1^2 y'(x)\left[1 + x^2 y'(x)\right] dx$$

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Solution:- Let $I[y] = \int_1^2 y'(1+x^2y')\,dx$

$$= \int_1^2 (y' + x^2y'^2)\,dx$$

Here $F = y' + x^2y'^2$

Since F is independent of y so E-L equation reduces to $\frac{\partial F}{\partial y'} = \text{constant}$

$$\frac{\partial}{\partial y'}[y' + x^2y'^2] = c$$

$$1 + 2x^2y' = c$$

$$2x^2y' = c - 1$$

$$y' = \frac{c-1}{2x^2}$$

integrating on both sides, we get

$$\int dy = \frac{c-1}{2}\int \frac{1}{x^2}\cdot dx$$

$$y = \frac{c-1}{2}\left[\frac{-1}{x}\right] + d \quad (\because d \text{ is constant})$$

$$y = \frac{1-c}{2x} + d$$

At A(1,3), we have $3 = \frac{1-c}{2(1)} + d$

$$6 = 1 - c + 2d$$

$$2d - c = 5 \quad \text{---(i)}$$

At B(2,5) we have

$$5 = \frac{1-c}{4} + d$$

$$\therefore 20 = 1 - c + 4d$$

$$4d - c = 19 \quad \text{---(ii)}$$

From (i) and (ii)

$$c = 9 \; ; \; d = 7$$

So $y = \frac{1-9}{2x} + 7 = 7 - \frac{4}{x}$

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Question 6 :- Find the curve that extremizes the functional $I = \int_0^{\pi/2} (y'^2 - y^2)\, dx$ with end points conditions $y(0) = 0$, $y(\pi/2) = 1$

Solution :- let

$$I = \int_0^{\pi/2} (y'^2 - y^2)\, dx$$

Here $F = y'^2 - y^2$

By E-L equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

$$\frac{\partial}{\partial y}[y'^2 - y^2] - \frac{d}{dx}\left[\frac{\partial}{\partial y'}(y'^2 - y^2)\right] = 0$$

$$-2y - \frac{d}{dx}(2y') = 0$$

$$-2y - 2y'' = 0$$

$$\Rightarrow y'' + y = 0$$

$$(D^2 + 1)\, y = 0$$

For non-trivial solution $y \neq 0$.

So $D^2 + 1 = 0$

$$D = \pm i$$

$$y(x) = A\cos x + B\sin x$$

where A and B are constants

$$y(0) = 0 \Rightarrow A = 0$$

$$y(\pi/2) = 1 \Rightarrow B = 1$$

Hence the requised extremal is

$$y(x) = \sin x$$

Written by
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A/2010, A/2009

Question 7 :- Find the extremal for the function. $I[y] = \int_{A(0,0)}^{B(\frac{\pi}{2},\frac{\pi}{2})} (y'^2 + 2xy - y^2)\,dx$

Solution :- Here

$$F = y'^2 + 2xy - y^2$$

By E-L equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

$$(2x - 2y) - \frac{d}{dx}(2y') = 0$$

$$2x - 2y - 2y'' = 0$$

$$y'' + y = x \quad \text{—— (1)}$$

Consider the Homogenous equation For Complimentry solution

$$y'' + y = 0$$

$$(D^2 + 1)y = 0 \quad \text{—— (2)}$$

For non-trivial solution $y \neq 0$

$$D^2 + 1 = 0 \Rightarrow D = \pm i$$

$$y_c = A\cos x + B\sin x$$

For particular integral $y_p = \frac{1}{F(D)} F(x)$

From (1) $\Rightarrow$ Here $F(D) = D^2 + 1$, $F(x) = x$

So

$$y_p = \frac{1}{(1 + D^2)} x$$

$$y_p = (1 + D^2)^{-1}(x)$$

$$y_p = \left[1 - D^2 + \frac{(-1)(-1)}{2!} D^4 + \text{——}\right] x$$

$$y_p = x - D^2(x) + \frac{-2}{2!} D^4(x) + \text{——}$$

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$$y_p = x - 0 + 0$$

$$\boxed{y_p = x}$$

Hence General solution is

$$y = y_c + y_p$$

$$y = A\cos x + B\sin x + x$$

At $A(0,0)$, we have

$$0 = A(1) + 0 + 0 \Rightarrow \boxed{A = 0}$$

At $B(\frac{\pi}{2}, \frac{\pi}{2})$, we have

$$\frac{\pi}{2} = 0 + B(1) + \frac{\pi}{2}$$

$$\boxed{B = 0}$$

Hence the extremal curve for the given functional

$$\boxed{y = x}$$

A/2008

Question 8:- Find the extremal of functional

$$I[y] = \int_0^{\pi} (4y\cos x + y'^2 - y^2)\,dx$$

$$y(0) = 0, \quad y(\pi) = 0$$

Solution:-

Here $F = 4y\cos x + y'^2 - y^2$

By E-L equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

$$-2y + 4\cos x - \frac{d}{dx}(2y') = 0$$

$$-2y + 4\cos x - 2y'' = 0$$

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$$y'' + y = 2\cos x \quad \text{— (1)}$$

For Complimentry solution,

Let

$$y'' + y = 0$$

$$(D^2+1)y = 0$$

For non-trivial solution $y \neq 0$

$$D^2+1 = 0$$

$$D = \pm i$$

$$y_c = A\cos x + B\sin x$$

Now we find particular integral

$$\therefore \quad y_p = \frac{1}{F(D)} \cdot F(x)$$

Here $F(D) = D^2+1$ and $F(x) = 2\cos x$

$$y_p = \frac{1}{D^2+1} \cdot 2\cos x$$

$$= 2\,\frac{1}{-(1)^2+1} \cdot \cos x$$

Failure Case

$$= \frac{2}{2D} \cdot x \cos x$$

$$= x\,\frac{1}{D}(\cos x) \qquad \left(\because \frac{1}{D} = \text{integration}\right)$$

$$y_p = x \sin x$$

Hence General solution is $y = y_c + y_p$

$$y = A\cos x + B\sin x + x\sin x$$

Now using the given conditions

$$y(0) = 0 \Rightarrow 0 = A + 0 + 0 \Rightarrow \boxed{A = 0}$$

$$y(\pi) = 0 \Rightarrow 0 = -A + 0 + 0 \Rightarrow A = 0$$

Hence $y = (B+x)\sin x$ where B is arbitrary constant.

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Question 9:- $I[y]=\int_0^1 (y'^2+4y^2)\,dx$

$y(0)=e^2$; $y(1)=1$

Solution:-

Here $F = y'^2 + 4y^2$

By E-L equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0$

$$8y - \frac{d}{dx}(2y') = 0$$

$$8y - 2y'' = 0$$

$$y'' - 4y = 0$$

$$(D^2-4)y = 0$$

For non-trivial solution $y \neq 0$

$$D^2 - 4 = 0$$

$$D = \pm 2$$

$$y = c_1 e^{2x} + c_2 e^{-2x}$$

At $y(0) = e^2$

$$c_1 + c_2 = e^2 \quad \text{---(i)}$$

At $y(1)=1$ $\quad c_1 e^2 + c_2 e^{-2} = 1$ ---(ii)

From (i) $c_2 = e^2 - c_1$ put in (ii)

$$1 = c_1 e^2 + (e^2 - c_1)e^{-2}$$

$$1 = c_1 e^2 + 1 - c_1 e^{-2}$$

$$c_1(e^2 - e^{-2}) = 0$$

As $e^2 - e^{-2} \neq 0$ Then for $\boxed{c_1 = 0}$

From (i)

$$\boxed{c_2 = e^2}$$

Hence

$$y = e^2 e^{-2x}$$

$$y = e^{2(1-x)}$$

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Question 10:- $I[y]=\int_{x_1}^{x_2}(y^2+y'^2+2ye^x)dx$

$y(x_1)=y_1$, $y(x_2)=y_2$

Solution:-

Here $F = y^2-y'^2+2ye^x$

By E-L equation $\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0$

$$2y+2e^x-\frac{d}{dx}(2y')=0$$

$$2y+2e^x-2y''=0$$

$$2y''=2y+2e^x$$

$$y''-y=e^x$$

For complementry solution

$$y''-y=0$$

$$(D^2-1)y=0$$

But $y\neq 0$ So $D^2-1=0 \Rightarrow D=\pm 1$

$$y_c=c_1e^x+c_2e^{-x}$$

Now we find particular integral

$$y_p=\frac{e^x}{D^2-1}$$

Co-efficient of $x=1$, put $D=1$ Then it become falicue case.

Then for

$$y_p=x\,\frac{1}{2D}e^x$$

$$y_p=\frac{xe^x}{2}$$

Hence General solution is

$$y=y_c+y_p$$

$$y=c_1e^{-x}+c_2e^{-x}+\frac{xe^x}{2}$$

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Question 11 :- Find the extremal of functional

$$I[y] = \int_0^{\pi/2} (y^2 - y'^2 - 8y\cosh x)\,dx$$

$y(0) = 2$, $y(\pi/2) = 2\cosh(\pi/2)$

Solution :- Here $F = y^2 - y'^2 - 8y\cosh x$

By E-L equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

$$2y - 8\cosh x - \frac{d}{dx}(-2y') = 0$$

$$2y - 8\cosh x + 2y'' = 0$$

$$y'' + y = 4\cosh x$$

$$y'' + y = 4\left[\frac{e^x + e^{-x}}{2}\right] = 2(e^x + e^{-x})$$

For C-F $\Rightarrow (D^2+1)y = 0$

But $y \neq 0$ $\quad D^2 + 1 = 0 \Rightarrow D = \pm i$

$$y_c = A\cos x + B\sin x$$

For P-I

$$y_p = \frac{1}{D^2+1} 2(e^x + e^{-x}) = \frac{2e^x}{D^2+1} + \frac{2e^{-x}}{D^2+1}$$

$$= \frac{2e^x}{2} + \frac{2e^{-x}}{2}$$

$$y_p = e^x + e^{-x}$$

Hence General solution is $y = y_c + y_p$

$$y = A\cos x + B\sin x + e^x + e^{-x}$$

At $y(0) = 2$, we get

$$2 = A + 2(1) \Rightarrow \boxed{A = 0}$$

At $y(\pi/2) = 2\cosh(\pi/2)$, we get

$$2\cosh(\pi/2) = B\sinh(\pi/2) + 2\cosh(\pi/2)$$

$$0 = B\sinh(\pi/2)$$

As $\sinh(\pi/2) \neq 0$ So $B = 0$

Hence $y = 2\cosh x$

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A/2016, A/2014, A/2012

Question 12:- Find the extremal of the functional

$$I[y] = \int^{\pi/2} (y''^2 - y^2 + x^2)\, dx$$

Subject to B.C's

$y(0) = 1$, $y(\pi/2) = 0$, $y'(0) = 0$, $y'(\pi/2) = 1$

Solution:- Here $F = y''^2 - y^2 + x^2$

The extremal $y = y(x)$ is obtained by solving the E-L equation

$$\frac{\partial F}{\partial y} + (-1)\frac{\partial F}{\partial y'} + (-1)^2 \frac{\partial F}{\partial y''} = 0$$

$$-2y + 0 + 2y^{(4)} = 0$$

$$y^{(4)} - y = 0$$

$$(D^4 - 1)y = 0$$

For non-trivial solution

$$D^4 - 1 = 0$$

$$(D^2 - 1)(D^2 + 1) = 0$$

$$(D-1)(D+1)(D^2+1) = 0$$

$$D = 1, -1, \pm i$$

$$y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x \quad \text{---(1)}$$

$$y' = c_1 e^x - c_2 e^{-x} - c_3 \sin x + c_4 \cos x \quad \text{---(2)}$$

using B.C's

$y(0) = 1 \Rightarrow 1 = c_1 + c_2 + c_3$ ---(i)

$y(\pi/2) = 0 \Rightarrow 0 = c_1 e^{\pi/2} + c_2 e^{-\pi/2} + c_4$ ---(ii)

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$y'(0)=0 \Rightarrow 0 = c_1 - c_2 + c_4$ —(iii)

$y'(\pi/2) \Rightarrow 1 = c_1 e^{\pi/2} - c_2 e^{-\pi/2} - c_3$ —(iv)

Adding (i) and (ii)

$2 = c_1(1+e^{\pi/2}) - c_2(1-e^{-\pi/2})$ —(v)

Subtracting (ii) from (iii)

$0 = c_1(1-e^{\pi/2}) - c_2(1+e^{-\pi/2})$ —(vi)

(v) $\Rightarrow$

$0 = c_1(1+e^{\pi/2}) + c_2(1-e^{-\pi/2}) - 2$ —(vii)

Solving (vi) and (vii), simultaneously, we have

$$\begin{array}{cc|c} 1-e^{\pi/2} & -(1+e^{-\pi/2}) & 0 \\ 1+e^{\pi/2} & 1-e^{-\pi/2} & -2 \end{array} \qquad \begin{array}{cc|c} 1-e^{\pi/2} & -(1+e^{-\pi/2}) & 0 \\ 1+e^{\pi/2} & 1-e^{-\pi/2} & -2 \end{array}$$

$$\frac{c_1}{2(1+e^{-\pi/2})} = \frac{c_2}{2(1-e^{\pi/2})} = \frac{1}{(1-e^{\pi/2})(1-e^{-\pi/2}) + (1+e^{\pi/2})(1+e^{-\pi/2})}$$

$\Rightarrow c_1 = \frac{1}{4}[2(1+e^{-\pi/2})]$

$c_1 = \frac{1}{2}[1+e^{-\pi/2}]$

$c_2 = \frac{1}{4}[2(1-e^{\pi/2})]$

$c_2 = \frac{1}{2}[1-e^{\pi/2}]$

(i) $\Rightarrow c_3 = 1 - \frac{1}{2}(1+e^{-\pi/2}) - \frac{1}{2}(1-e^{\pi/2})$

$= 1 - \frac{1}{2} - \frac{1}{2}e^{-\pi/2} - \frac{1}{2} + \frac{1}{2}e^{\pi/2}$

$= \frac{1}{2}[e^{\pi/2} - e^{-\pi/2}]$

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(iii) $\Rightarrow$ $C_4 = C_2 - C_1$

$$= \frac{1}{2}\left[(1-e^{\pi/2}) - (1+e^{-\pi/2})\right]$$

$$= -\frac{1}{2}\left[e^{\pi/2} + e^{-\pi/2}\right]$$

Hence

$$y = \frac{1}{2}(1+e^{-\pi/2})e^{x} + \frac{1}{2}(1-e^{\pi/2})e^{-x}$$

$$+\frac{1}{2}(e^{\pi/2}-e^{-\pi/2})\cos x - \frac{1}{2}(e^{\pi/2}+e^{-\pi/2})\sin x$$

A/2013 . A/2012

Question 13:- Find the extremal for

$$I = \int^{\pi/2} (y'^2 + z'^2 + 2yz)\,dx$$

$y(0)=0$, $y(\pi/2)=1$, $z(0)=0$, $z(\pi/2)=-1$

Solution:- Since there are two unknown functions extremal curves y and z, there will be a pair of Euler-Lagrange : then there corresponding E-L equation is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad , \quad \frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) = 0$$

Here $F = y'^2 + z'^2 + 2yz$

$$2z - \frac{d}{dx}(2y') = 0 \quad ; \quad 2y - \frac{d}{dx}(2z') = 0$$

$$2z - 2y'' = 0 \quad ; \quad 2y - 2z'' = 0$$

$$y'' - z = 0 \text{ —(i)} \quad ; \quad z'' - y = 0 \text{ —(ii)}$$

From (ii)

$$y = z''$$

put in (i)

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$$z^{(4)} - z = 0$$

$$(D^4 - 1) z = 0$$

For non-trivial solution

$$D^4 - 1 = 0$$

$$(D^2+1)(D^2-1) = 0$$

$$(D^2+1)(D+1)(D-1) = 0$$

$$D = \pm 2, 1, -1$$

So $y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$ — (A)

Also from (ii)

$$z = y''$$

$$z = \frac{d}{dx}\left[c_1 e^x + c_2 e^{-x} - c_3 \sin x + c_4 \cos x\right]$$

$$z = c_1 e^x - c_2 e^{-x} - c_3 \cos x - c_4 \sin x \quad \text{— (B)}$$

using the given conditions

$y(0) = 0 \Rightarrow 0 = c_1 + c_2 + c_3$ — (i)

$y(\pi/2) = 1 \Rightarrow 1 = c_1 e^{\pi/2} + c_2 e^{-\pi/2} + c_4$ — (ii)

$z(0) = 0 \Rightarrow 0 = c_1 + c_2 - c_3$ — (iii)

$z(\pi/2) = -1 \Rightarrow -1 = c_1 e^{\pi/2} + c_2 e^{-\pi/2} - c_4$ — (iv)

Adding (i) and (iii)

$$c_1 + c_2 + c_3 = 0$$

$$c_1 + c_2 - c_3 = 0$$

$$2c_1 + 2c_2 = 0$$

$$c_2 = -c_1 \quad \text{— (v)}$$

Subtracting (ii) and (iv), we get

$$\boxed{c_4 = 1}$$

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Subtracting (i) and (ii), we get

$$c_3 = 0$$

Adding (i) and (iii), we get

$$c_1 e^{\pi/2} + c_2 e^{-\pi/2} = 0$$

$$c_1 \left[e^{\pi/2} + e^{-\pi/2} \right] = 0$$

As $e^{\pi/2} \neq e^{-\pi/2}$ So $\boxed{c_1 = 0}$

(v) $\Rightarrow$ $\boxed{c_2 = 0}$

Hence the required extremal are

$$y = \sin x \qquad z = \sin x$$

Question 14 :- Find the extremal for

$$I[y] = \int_0^1 (yy' + y''^2)\, dx$$

$y(0) = 0$, $y'(0) = 1$, $y(1) = 2$, $y'(1) = 4$

Solution :- $F = yy' + y''^2$

For Extremal curve $y(x)$ satisfies EL-Equation

$$\frac{\partial F}{\partial y} + (-1)\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + (-1)^2 \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0$$

$$y' - \frac{d}{dx}(y) + \frac{d^2}{dx^2}(2y'') = 0$$

$$y' - y' + 2y^{(4)} = 0$$

$$y^{(4)} = 0$$

integrating $y^{(3)} = c$

Again integrating $y'' = cx + d$

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$$y' = \frac{cx^2}{2} + dx + e$$

$$y = \frac{cx^3}{6} + \frac{dx^2}{2} + ex + f$$

Now applying given conditions

$y(0) = 0 \Rightarrow \boxed{f = 0}$

$y(1) = 2 \Rightarrow 2 = \frac{c}{6} + \frac{d}{2} + e$ —— (i)

$y'(1) = 4 \Rightarrow 4 = \frac{c}{2} + d + e$ —— (ii)

$y'(0) = 1 \Rightarrow \boxed{e = 1}$

By (i) & (2) $\boxed{c = 6}$, $\boxed{d = 0}$

Hence $y = x^3 + x$

S/2009

Question 15 :- Find the extremal for the functional

$$I[y, z] = \int_1^2 (y'^2 + z'^2 + y^2)\, dx$$

$y(1) = z(2) = 1$, $y(2) = 2$, $z(1) = 0$

Solution :- Here $F = y'^2 + z'^2 + y^2$

The E-L equation are

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \text{—— (i)}$$

$$\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) = 0 \quad \text{—— (ii)}$$

(i) $\Rightarrow 2y - \frac{d}{dx}(2y') = 0$

$2y - 2y'' = 0$

$y'' - y = 0$

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$$(D^2-1)y=0$$

For non-trivial solution $y \neq 0$

$$D^2-1=0$$

$$D=\pm 1$$

$$y(x)=c_1e^x+c_2e^{-x}$$

$$y(1)=1 \Rightarrow 1=c_1e+c_2e^{-1}$$

$$c_1e+c_2e^{-1}-1=0 \quad \text{---(iii)}$$

$$y(2)=2 \Rightarrow 2=c_1e^2+c_2e^{-2}$$

$$c_1e^2+c_2e^{-2}-2=0 \quad \text{---(iv)}$$

Solving (iii) and (iv) simultaneously we have

$$\begin{array}{c|c|c|c} e & e^{-1} & -1 & e \quad e^{-1} \\ e^2 & e^{-2} & -2 & e^2 \quad e^{-2} \end{array} \quad \begin{array}{|c} -1 \\ -2 \end{array}$$

$$\frac{c_1}{-2e^{-1}+e^{-2}} = \frac{c_2}{-e^2+2e} = \frac{1}{e^{-1}-e}$$

$$c_1=\frac{-2e^{-1}+e^{-2}}{e^{-1}+e}, \quad c_2=\frac{-e^2+2e}{e^{-1}-e}$$

$$c_1=\frac{1-2e}{e-e^3}; \quad c_2=\frac{2e^2-e^3}{1-e^2}$$

Hence $$y=\left(\frac{1-2e}{e-e^3}\right)e^x+\left(\frac{2e^2-e^3}{1-e^2}\right)e^{-x}$$

Next (ii) $\Rightarrow$ Since z is absent in F

So $$\frac{\partial F}{\partial z}=\text{Constant}$$

$$2z'=c$$

$$z'=\frac{c}{2}$$

$$z=\frac{cx}{2}+d$$

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$z(2) = 1 \Rightarrow 1 = c + d$ —— (v)

$z(1) = 0 \Rightarrow 0 = \frac{c}{2} + d \Rightarrow c + 2d = 0$ —— (vi)

From (v) & (vi)

$c = 2, \quad d = -1$

Hence $z = x - 1$

A/2009

Question 16 :- Find the extremal curve for the functional

$$I[y, z] = \int_0^{\pi/2} (2yz - 2y^2 + y'^2 - z'^2)\, dx$$

$y(0) = 0 \Rightarrow z(0) =$ ..... $y(\pi/2) = 1, \; z(\pi/2) = -1$

Solution :- Here $F = 2yz - 2y^2 + y'^2 - z'^2$

The corresponding EL-Equation are

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \text{—— (i)}$$

$$\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) = 0 \quad \text{—— (ii)}$$

(i) $\Rightarrow (2z - 4y) - \frac{d}{dx}(2y') = 0$

$2z - 4y - 2y'' = 0$

$y'' + 2y - z = 0$ —— (iii)

(ii) $\Rightarrow z = y' + 2y$

$z'' = y^{(4)} + 2y''$

$-y = y^{(iv)} + 2 \cdot y''$

$y^{(4)} + 2y'' + y = 0$

(ii) $\Rightarrow 2y - \frac{d}{dx}(-2z') = 0$

$2y + 2z'' = 0$

$z'' + y = 0$ —— (iv)

$(\because z'' = -y)$

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characteristic equation is

$$(D^4 + 2D^2 + 1)y = 0$$

$$[(D^2+1)^2]y = 0$$

For non-trivial solution $y \neq 0$

$$(D^2+1)^2 = 0$$

$$(D^2+1)(D^2+1) = 0$$

$$D = \pm i, \pm i$$

$$y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x \quad \text{(A)}$$

$$y(0) = 0 \Rightarrow c_1 = 0$$

$$y(\pi/2) = 1 \Rightarrow 1 = c_3 + \frac{\pi}{2} c_4 \quad \text{(V)}$$

Hence

$$y = c_2 x \cos x + (c_3 + c_4 x)\sin x$$

$$y = c_2 x \cos x + c_3 \sin x + c_4 x \sin x$$

Now

$$z = y'' + 2y \quad \text{(VI)}$$

$$y' = c_2\{-x\sin x + \cos x\} + c_3 \cos x + c_4\{x\cos x + \sin x\}$$

$$y' = -c_2\{+x\sin x\} + c_2\cos x + c_3\cos x + c_4 x\cos x + c_4\sin x$$

$$y'' = -c_2(x\cos x + \sin x) - c_2\sin x - c_3\sin x + c_4(-x\sin x + \cos x) + c_4\cos x$$

$$= -c_2 x\cos x - c_2\sin x - c_2\sin x - c_3\sin x - c_4 x\sin x + c_4\cos x + c_4\cos x$$

$$= -c_2 x\cos x - 2c_2\sin x - c_3\sin x - c_4 x\sin x + 2c_4\cos x$$

Putting in (VI)

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(366)

370

$z = -c_2 x\cos x - 2c_2 \sin x - c_3 \sin x - c_4 x \sin x$

$+2c_4 \cos x + 2c_2 x\cos x + 2c_3 \sin x + 2c_4 x \sin x$

$\Rightarrow z = c_2 x\cos x + c_4 x \sin x + c_3 \sin x - 2c_2 \sin x + 2c_4 \cos x$

$z(0) = 0 \Rightarrow c_4 = 0$

$z(\pi/2) = -1 \Rightarrow -1 = c_3 - 2c_2 \Rightarrow c_3 - 2c_2 = -1$

$\Psi \Rightarrow$ $c_3 = 1$ $\Rightarrow$ $c_2 = 1$

Hence the required external are

$y = x\cos x + \sin x$

$z = x\cos x - \sin x$

Best of Luck

Labeeb Ahmad

From

Shasaq pur Shaorif

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